Some Further Remarks on Total Boundness

A. Hereditarity.

Lemma. If A is d-totally bounded and $B \subseteq A$, then B is d-totally bounded.

Proof. For $\varepsilon > 0$, there exists a $\{O_d(x_1,\varepsilon), \dots, O_d(x_n,\varepsilon)\}$ such that $x_1,\dots,x_n \in X$ and $A \subseteq \bigcup_{i=1}^{n} O_d(x_i,\varepsilon)$. But $B \subseteq A$ hence $B \subseteq \bigcup_{i=1}^{n} O_d(x_i,\varepsilon)$. Hence there exists an finite ε-cover of open balls for B and since ε is arbitrary the result follows. (Notice that it must be that for any $\varepsilon>0$, $N(\varepsilon,d,B) \leq N(\varepsilon,d,A)$ - why?). □

The dual result simply says that if $B \subseteq A$ and B is not d-totally bounded (i.e. $\exists\, \varepsilon > 0$ such that A cannot be covered by any finite collection of ε-open balls of X) then A is also not d-totally bounded.

B. Total Boundness in $(\mathbb{R}, d_u)$.

Suppose that $A = [x-\delta, x+\delta]$, $x \in \mathbb{R}$, $\delta > 0$. Notice that given $\varepsilon > 0$, A can be covered by $\lfloor \delta/\varepsilon \rfloor$ closed balls of radius ε, the $[x-\delta, x-\delta+2\varepsilon] = O_{d_u}[x+\varepsilon,\varepsilon]$, $[x-\delta+2\varepsilon, x-\delta+4\varepsilon] = O_{d_u}[x+2\varepsilon,\varepsilon], \dots, [x-\delta+(\lfloor \delta/\varepsilon \rfloor -1)2\varepsilon,\ x-\delta+\lfloor \delta/\varepsilon \rfloor 2\varepsilon] = O_{d_u}[x-\delta+2\lfloor \tfrac{\delta}{\varepsilon} \rfloor -\varepsilon, \varepsilon]$, where $\lfloor x \rfloor :=$ the smallest $n \in \mathbb{N} : n \geq x$. Since ε is arbitrary this implies that A is d_u-totally bounded, by the equivalence of the definition for closed balls. But due to A. Hereditarity, this implies that any d_u-bounded subset of $\mathbb{R}$ is also d_u-totally bounded (why?), which then implies that the two notions coincide on $(\mathbb{R}, d_u)$ (why?). By elaborating more on the previous we can actually prove that if B is d_u-totally bounded then

$$N(\varepsilon, d_u, B) \sim \frac{c_B}{\varepsilon} \quad \left(\text{i.e. as } \varepsilon \to 0^+,\ \frac{N(\varepsilon,d_u,B)}{c_B/\varepsilon} \to 1\right) \text{ where}$$

$c_B > 0$ that depends on B. □

C. Comparison of Metrics

Lemma.

If d^*, d are well defined metrics on X and $d^* \leq c d$ for $c > 0$, then if A is d-totally bounded, it is also d^*-totally bounded.

Proof. Since $\forall \varepsilon > 0$ and $x \in X$, $O_d(x, \varepsilon/c) \subseteq O_{d^*}(x,\varepsilon)$, and if the $x_1, x_2, \dots, x_{n(\varepsilon)}$ define $\{O_d(x_1,\varepsilon/c), O_d(x_2,\varepsilon/c), \dots, O_d(x_{n(\varepsilon/c)}, \varepsilon/c)\}$, which is an ε/c-d-cover by open balls of A then $\{O_{d^*}(x_1,\varepsilon), O_{d^*}(x_2,\varepsilon), \dots, O_{d^*}(x_{n(\varepsilon/c)},\varepsilon)\}$ is necessarily an ε-d^* cover of open balls for A. $\square$

Exercise: Would the above hold if $d^* \leq f(d)$, where $f: \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing?

It is easy to see from the previous that $N(\varepsilon, d^*, A) \leq N(\varepsilon/c, d, A)$.

Corollary. If for $c^*, c > 0$, $c^* d \leq d^* \leq cd$ then A is d^*-totally bounded iff A is d-totally bounded.

Notice also that the previous imply that $N(\varepsilon, d^*, A) \leq N(\varepsilon/c, d, A)$ and
$N(\varepsilon, d, A) \leq N(c^*\varepsilon, d^*, A)$

hence $N(\varepsilon, d, A) \leq N(c^*\varepsilon, d, A) \leq N(\frac{c^*}{c}\varepsilon, d, A)$ (which is the corrected version of the system of inequalities that we have derived in the class).

Corollary. $A \subseteq \mathbb{R}^n$ is d^*-totally bounded iff it is d-totally bounded, where $d^*, d = d_I, d_{II}, d_{max}$. $\square$

D. Finite Products

Suppose that $X = \prod_{i=1}^{n} X_i$, (X_i, d_i) are metric spaces and consider $d_{\Pi max}$.

Lemma. $A = \prod_{i=1}^{n} A_i$, $A_i \subseteq X_i$, $i = 1, \dots, n$ is $d_{\Pi max}$-totally bounded iff A_i is d_i-totally bounded $\forall i = 1, \dots, n$. $\square$

Proof. Exercise. $\square$

Corollary. For $(\mathbb{R}^n, d_{max})$, A is d_{max}-totally bounded iff it is d_{max}-bounded.

Proof. ($\Rightarrow$) follows directly from the relation between the notions.

($\Leftarrow$) If A is d_{max}-bounded then $\exists x \in \mathbb{R}^n, \varepsilon > 0 : A \subseteq O_{d_{max}}(x, \varepsilon)$ $= \prod_{i=1}^{n}$ of closed intervals in $\mathbb{R}$ ($\Rightarrow$) Every such interval is d_u-bounded (why?) $\Rightarrow$ from the Lemma in B that every such interval is d_u-totally bounded $\Rightarrow$ from the previous lemma that $O_{d_{max}}[x,\varepsilon]$ is d_{max}-totally bounded (why?) $\Rightarrow$ from the hereditarity lemma that A is d_{max}-totally bounded. $\square$

Corollary. If $A \subseteq \mathbb{R}^n$ then A is d-bounded iff it is d^*-totally bounded, for $d, d^* = d_I, d_{II}, d_{max}$.

Proof. Combine the previous. $\square$

E. Metric Entropy

If (X,d) is totally bounded then the $\ln N(\varepsilon, d, X)$ are called Metric Entropy Numbers of X. They typically diverge to infinity as $\varepsilon \to 0$ and the rate provides information for the "complexity" of X allowing for comparisons between different spaces.

E.g. if $A \subseteq \mathbb{R}$, d_u-totally bounded then we have already seen that

$$\ln N(\varepsilon, d_u, A) \sim \ln C_A - \ln \varepsilon,$$ and we can

generalize it to that if $A \subseteq \mathbb{R}^n$, d^* totally bounded then

$$\ln N(\varepsilon, d^*, A) \sim \ln C_A^* - n \ln \varepsilon,$$ where C_A^* depends on d^* and A, for any $d^* = d_I, d_{II}, d_{max}$.

As a further example let $X_L = \{f : [0,1] \to \mathbb{R}, f(0) = 0, \forall x,y \in [0,1], |f(x) - f(y)| \leq L|x-y|\}$, for $L > 0$ and $d = d_{sup}$. We have already proven that X_L is d_{sup}-bounded, and it is possible to prove that it is d_{sup}-totally bounded with metric entropy

$$\ln N(\varepsilon, d_{sup}, X_L) \sim C_L/\varepsilon,$$ for $C_L > 0$ (that depends on L). Hence the latter is more "complicated" space than any A-d_I-totally bounded subset of $\mathbb{R}^n$ for any $n > 0$.

F. Application - Metric Entropy Integral

Remember that an $\mathbb{R}$ valued stochastic process over $\Theta\,(\neq\emptyset)$ is a collection of random variables $\{X_\theta : \Omega \to \mathbb{R},\ \theta\in\Theta\}$ given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under the Daniell-Kolmogorov Theorem this is simultaneously an appropriately measurable function $X : \Omega \to \mathbb{R}^\Theta = \{f : \Theta \to \mathbb{R}\}$. Remember also that X is Gaussian iff every fidi of X is a Normal distribution of appropriate dimension.

Question. When would it be possible that $X(\omega,\cdot) : \Theta \to \mathbb{R}$ is an appropriately continuous function of θ, for almost all $\omega\in\Omega$?

The following astonishing result implies that this may depend on the metric entropy properties of Θ when it is endowed with a metric that reflects probabilistic properties of X, when the latter is Gaussian.

Dudley's Metric Entropy Integral Theorem. Suppose that X is Gaussian. Endow Θ with the (pseudo-) metric $d^*(\theta_1,\theta_2) := \left[E\,(X_{\theta_1} - X_{\theta_2})^2\right]^{1/2}$. If (Θ, d^*) is totally bounded and for some $\delta>0$, $\int_0^\delta [\ln N(\varepsilon, d^*, \Theta)]^{1/2} d\varepsilon < +\infty$ then

$X(\omega,\cdot) : \Theta \to \mathbb{R}$ is "appropriately" continuous (w.r.t. d_u on $\mathbb{R}$ and d^* on Θ), for $\mathbb{P}$ almost all $\omega\in\Omega$. $\square$

Remarks:

1. If such δ exists, then actually $\int_0^{+\infty} [\ln N(\varepsilon, d^*, \Theta)]^{1/2} < +\infty$ (why?).

2. When we later on understand well the notion of continuity of a function between metric spaces we will construct example(s). $\square$