A Counterexample

The following is a counterexample of a function that fails to have the properties of a metric. It was briefly discussed during a break.

Let $X=\mathbb{R}$ and $d^*:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined as

$$d^*(x,y)=\begin{cases}|x-y|, & \text{if } (x,y)\neq(0,0)\\ 1, & \text{if } (x,y)=(0,0).\end{cases}$$

Obviously d^* does not satisfy separation, due to its definition at $(0,0)$. Furthermore, it does not satisfy the triangle inequality since for $x=y=0$, $z=1/3$,

$$d^*(x,y)=1>1/3+1/3=d^*(x,z)+d^*(z,y).$$

Hence d^* cannot be a metric.

1. If we define real intervals via d^* it is easy to see that any open interval centered at zero, with radius less than 1 does not contain zero, i.e. it has the form $(-\varepsilon,0)\cup(0,\varepsilon)$, for any $0<\varepsilon<1$, which would then imply that any eventually constant at zero sequence, e.g. $(0,0,\dots,0,\dots)$ does not "converge" to zero w.r.t. d^* (keep this in mind, as it will become clear in the following lectures), whereas a sequence of the form $(1,\frac{1}{2},\dots,\frac{1}{n+1},\dots)$ would "converge" to zero w.r.t. d^*.

2. d^* becomes a metric when we get rid of its pathology at zero, i.e. when we define it on $\mathbb{R}-\{0\}=(-\infty,0)\cup(0,+\infty)$.