For (X,d) a metric space, and $A \subseteq X$ a finitary property which A could satisfy w.r.t. d, concerns the possibility of A being a subset of a(n) (open) ball. This is a direct generalization of the notion of a bounded subset of $\mathbb{R}$ (w.r.t. the usual metric).

Definition. A is (d-) bounded iff $\exists x \in X, \varepsilon > 0$:

$$A \subseteq O_d(x,\varepsilon). \square$$

Remarks:

1. If A is (d-) bounded then ε is not unique. Obviously $A \subseteq O_d(x,\delta)$ $\forall \delta \geq \varepsilon$. (remember the inclusion lemma for balls)

2. The definition is equivalent to the one obtained if the demand for the existence of an open ball is substituted by an analogous concerning a closed (again due to the lemma of inclusions for balls)

3. $\emptyset$ is always bounded since $\emptyset \subseteq O_d(x,\varepsilon)$, $\forall x \in X, \varepsilon > 0$, d a metric (due to the lemma on the non-emptyness of balls)

4. A ball is always bounded. (obvious)

5. The definition enables the generalization of the concept of bounded function. Hence $f: Y \to X$ is (d-) bounded iff $f(Y) = \{x \in X : x = f(y), y \in Y\}$ is a (d-) bounded subset of X.

6. If A is (d-) bounded and $B \subseteq A \Rightarrow B$ is (d-) bounded. (State the contra-positive)

Definition. (X,d) is called bounded iff X is a $(d-)$ bounded subset of itself. □

Examples.

1. $\mathbb{R}$ with the usual metric is not bounded. For if it were there would exist $x \in \mathbb{R}$, $\varepsilon > 0$: $\mathbb{R} \subseteq (x-\varepsilon, x+\varepsilon)$ which is obviously absurd. □

2. $\mathbb{R}_- := \{x \in \mathbb{R}, x \leq 0\}$ with d_e (remember the concept of a metric subspace) is bounded. Consider $x = 0$, $\varepsilon \geq 1$, then $O_{d_e}(0,\varepsilon) = (-\infty, 0] = \mathbb{R}_-$. Analogously to the previous example $\mathbb{R}_-$ with the usual metric is not bounded. Hence the property crucially depends on d.

3. If X is finite, the (X,d) is bounded for any possible d. This is due to the face that for $\varepsilon := \max\limits_{y,x \in X} d(x,y) < +\infty$ since X is finite $X \subseteq O_d(x, k\varepsilon)$, $\forall x \in X$ and $k > 1$. This is due to the fact that $d(x,y) \leq \max\limits_{x,y \in X} d(x,y) = \varepsilon < k\varepsilon \Rightarrow y \in O_d(x, k\varepsilon)$, $\forall x \in X$.

4. (X, d_S) is bounded $\forall X \neq \phi$. [Discrete Spaces are always bounded!] This is due to the face that $\forall \varepsilon > 1$, $X \subseteq O_{d_S}(x,\varepsilon) = X$, $\forall x \in X$.

5. $(\mathbb{R}^k, d_*)$ where $d_* = d_A$ or d_{11} or d_{max} is not bounded (prove it!).

6. $A \subseteq B(Y, \mathbb{R})$ is (d_{sup}-) bounded iff $\sup_{f\in A} \sup_{x\in Y} |f(x)| < +\infty$.

This is due to the following facts: ($\Rightarrow$) if $\sup_{f\in A} \sup_{x\in Y} |f(x)| < +\infty$

then setting $M := \sup_{f\in A} \sup_{x\in Y} |f(x)|$ we have that ($0(x) := 0 \ \forall x\in Y$)

$f\in A \Rightarrow f \in O_{d_{sup}}[0, M]$ since $\sup_{x\in Y} |f(x) - 0(x)| = \sup_{x\in Y} |f(x)| \leq M$,

hence A is bounded. ($\Leftarrow$) if A is bounded then $\exists g \in B(Y, \mathbb{R})$,

$\varepsilon > 0$: $A \subseteq O_{d_{sup}}[g, \varepsilon]$. Set $\delta := d_{sup}(g, 0) \geq 0$. Then $A \subseteq$

$O_{d_{sup}}[0, \delta+\varepsilon]$, since $f\in A \Rightarrow f \in O_{d_{sup}}[g, \varepsilon] \Rightarrow \sup_{x\in Y} |f(x) - 0(x)|$

$\overset{\text{tr.in.}}{\leq} \sup_{x\in Y} |f(x) - g(x)| + \sup_{x\in Y} |g(x) - 0(x)| \leq \varepsilon + \delta \Rightarrow f \in O_{d_{sup}}[0, \delta+\varepsilon]$.

Hence $\sup_{f\in A} \sup_{x\in Y} |f(x)| = \sup_{f\in A} \sup_{x\in Y} |f(x) - 0(x)| \leq \delta + \varepsilon < +\infty$. A

(d_{sup}-) bounded set is also called uniformly bounded. For

a counter-example consider, $Y = [0,1]$, $A = \{ f_n : [0,1] \to \mathbb{R},\ f_n(x) = nx,$

$n \in \mathbb{N} \}$. Notice that $\sup_{x\in[0,1]} |f_n(x)| = n < +\infty \Rightarrow f_n \in B([0,1], \mathbb{R})$

$\forall n \in \mathbb{N}$, but $\sup_{f\in A} \sup_{x\in[0,1]} |f(x)| = \sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} |nx| = \sup_{n\in\mathbb{N}} n = +\infty$

hence A is not uniformly bounded. (As a counter-example

consider also the relevant part of example 5 - explain!)

For an example, let $0<c<1$, $Y=[c,1]$, $A=\{f_n:[c,1]\to\mathbb{R},\ f_n(x)=\frac{x}{n+1},\ n\in\mathbb{N}\}$. Notice that $\sup\limits_{f\in A}\sup\limits_{x\in[c,1]}|f(x)| =$

$= \sup\limits_{n\in\mathbb{N}}\sup\limits_{x\in[c,1]}\left|\frac{x}{n+1}\right| = \sup\limits_{n\in\mathbb{N}}\frac{1}{n+1} = 1<+\infty$. Hence A is a uniformly bounded subset of $B([c,1],\mathbb{R})$. $\square$

Lemma 1. Suppose that $A\subseteq\mathbb{R}^k$ is d_{II}-bounded. Then it is also d_I bounded.

Proof. From a previous result (explain) we have that $d_I\le d_{II}$ as functions. Since A is d_I-bounded, then $\exists x\in X, \varepsilon>0$:
$A\subseteq O_{d_{II}}(x,\varepsilon) \Rightarrow \big(\text{if } y\in A \Rightarrow d_{II}(x,y)<\varepsilon \Rightarrow d_I(x,y)<\varepsilon \Rightarrow y\in O_{d_{II}}(x,\varepsilon)\big)$

$\Rightarrow A\subseteq O_{d_I}(x,\varepsilon)$. $\square$

Lemma 2. In the context of Lemma 1, if A is d_I-bounded then it is d_{max}-bounded.

Proof. Similar to the proof of Lemma 1, via the use of the (functional) inequality $d_{max}\le d_I$, derived in a previous paragraph (explain and fill in the details). $\square$

The previous lemmata are examples of how relations between metrics represent relations between properties of the relevant spaces.

Lemma 3. Suppose that I is finite, and A_i are d_i-bounded subsets of X_i, $i\in I$. Then, $\prod\limits_{i\in I} A_i$ is d_*-bounded for $d_* = d_{\Pi_I}$ or $d_{\Pi_{II}}$ or $d_{\Pi_{max}}$. The converse also holds.

Proof. (Direct) A_i is d_i-bounded $\forall i \in \mathcal{I} \Leftrightarrow \exists x_i \in X_i, \varepsilon_i > 0$:

$\left(A_i \subseteq O_{d_i}(x_i, \varepsilon_i) \ \forall i \in \mathcal{I}\right) \Leftrightarrow \left(y_i \in A_i \Rightarrow d_i(x_i, y_i) < \varepsilon_i \ \forall i \in \mathcal{I}\right)$

$\Rightarrow \left(y_i \in A_i \Rightarrow d_i(x_i, y_i) < \varepsilon := \max_{i \in \mathcal{I}} \varepsilon_i < +\infty \text{ (since } \mathcal{I} \text{ is finite)}\right)$

$$\underset{\substack{x := (x_i)_{i\in\mathcal{I}} \\ y := (y_i)_{i\in\mathcal{I}} \\ x, y \in \prod_{i\in\mathcal{I}} A_i \\ k := \#\mathcal{I}}}{\Longrightarrow} \left(y \in \prod_{i \in \mathcal{I}} A_i \Rightarrow \begin{cases} d_{\Pi_{max}}(x,y) < \varepsilon \\ d_{\Pi_{\mathcal{I}}}(x,y) < \sqrt{k}\,\varepsilon \\ d_{\Pi_{11}}(x,y) < k\varepsilon \end{cases} \Leftrightarrow \begin{cases} A \subseteq O_{d_{\Pi_{max}}}(x, \varepsilon) \\ A \subseteq O_{d_{\Pi_{\mathcal{I}}}}(x, \sqrt{k}\,\varepsilon) \\ A \subseteq O_{d_{\Pi_{11}}}(x, k\varepsilon) \end{cases} \right)$$

$\Rightarrow A$ is d_*-bounded.

((Converse)). Suppose that $\prod_{i\in\mathcal{I}} A_i$ $d_{\Pi_{max}}$-bounded.

Then $\exists x \in \prod_{i\in\mathcal{I}} X_i, \varepsilon > 0$: $\prod_{i\in\mathcal{I}} A_i \subseteq O_{d_{\Pi_{max}}}(x, \varepsilon) \Leftrightarrow$

(if $y \in \prod_{i\in\mathcal{I}} A_i \Rightarrow d_{\Pi_{max}}(x,y) < \varepsilon \Rightarrow d_i(x_i, y_i) < \varepsilon \ \forall i \in \mathcal{I}$

$y_i \in O_{d_i}(x_i, \varepsilon) \ \forall i \in \mathcal{I} \Leftrightarrow A_i \subseteq O_{d_i}(x_i \cdot \cdot) \ \forall i \in \mathcal{I}$ [Provide the details for the other two product metrics]. $\square$

Corollary. The converses of Lemmata 1-2 also hold.

Proof. [Provide the details]. $\square$

Hence when $\mathcal{I}$ is finite, then d_*-boundness is equivaleast for d_i-boundness of the factor sets.

The notes are in a state of perpetual correction. They do not substitute the lectures.
Please report any typos to stelios@aueb.gr or the course's e-class.]