Athens University of Economics and Business
Department of Economics

Postgraduate Program - Master's in Economic Theory
Course: Mathematical Analysis (Mathematics II)
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Semester: Spring 2016-2017

12-05-2017

EXERCISES 5

Contractions, the Banach Fixed Point Theorem, and extensions

1. Let $x_1 = \sqrt{2}$ and for $n \geq 1$ let $x_{n+1} = \sqrt{2 + \sqrt{x_n}}$. Use Banach's FPT to show that (x_n) converges to a root of the equation $x^4 - 4x^2 - x + 4 = 0$ lying between $\sqrt{3}$ and 2. *(exercise 17.10 in Sutherland's "Introduction to Metric and Topological Spaces").*

2. Define $f : (0,1/4) \to (0,1/4)$ by $f(x) = x^2$. Show that f is a contraction that has no fixed point. *(exercise 17.11 in Sutherland's "Introduction to Metric and Topological Spaces").*

3. The Banach Fixed Point Property: Every contraction $T \in S^S$, where S is a nonempty closed subset of a metric space X has a fixed point.

Prove that a metric space is complete if and only if it has the Banach Fixed Point Property.

4. Consider a system of linear equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $k \times k$ known and invertible matrix $\mathbf{x}$ is a $k \times 1$ vector of unknowns and $\mathbf{b}$ is a known $k \times 1$ vector. The system has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Construct a function f whose fixed point is the above solution. Examine under which conditions on $\mathbf{A}$, instead of inverting it to get the solution, we could obtain it by iterating the function f . *(example 10.10.8. from Sercoid's book).*

5. Define $f:[1,\infty)\to[1,\infty)$ by $f(x)=x+x^{-1}$. Show that $[1,\infty)$ is complete under the usual metric, and $|f(x)-f(y)|<|x-y|$ for any distinct $x,y\in[1,\infty)$, yet f has no fixed point. *(exercise 17.12 in Sutherland's "Introduction to Metric and Topological Spaces").* **Note**: this is a preparation for Edelstein's Fixed Point Theorem.

6. Edelstein's Fixed Point Theorem. A self-map T on a metric space (X,d) is said to be a **pseudo-contraction** if $d(T(x),T(y))<d(x,y)$ for all distinct $x,y\in X$. Prove that if X is **compact,** the pseudo-contraction T has a unique fixed point *(exercise 17.15 in Sutherland's "Introduction to Metric and Topological Spaces". Also Exercise C. 50(a) in Efe Ok's book).*

7. The Central Limit Theorem (CLT) as a Fixed Point Theorem.

Study the following paper: **Trotter, H. F. (1959). An elementary proof of the central limit theorem.** ***Archiv der Mathematik*, *10*(1), 226-234.**

It uses the concept of contraction and fixed point theory to prove the basic Central Limit Theorem, essentially showing that in the space of statistical distribution functions the normal distribution emerges as a fixed point for the distribution of the sum of independent random variables.

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