Athens University of Economics and Business
Department of Economics

Postgraduate Program - Master's in Economic Theory
Course: Mathematical Analysis (Mathematics II)
Prof: Stelios Arvanitis
TA: Alecos Papadopoulos

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EXERCISES 4

Continuity, Convergence, Completeness

1. A convex function is continuous. Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function, namely that for $\lambda\in[0,1]$ and $x_0,x_1\in\mathbb{R}$ we have $f\left(\lambda x_0+(1-\lambda)x_1\right)\ \le\ \lambda f\left(x_0\right)+(1-\lambda)f\left(x_1\right)$. Prove that f is continuous.

2. The composition of two continuous functions is continuous. For any metric spaces X,Y,Z let $f:X\to Y$ and $g:f\left(X\right)\to Z$ be continuous functions. Then their composition $h:=g\circ f$, $h(x)=g\left(f(x)\right)$ is continuous on X .

3. Construct a metric on $\mathbb{R}$ in which the sequence $(1/n)$ of inverses of natural numbers converges to a limit other than 0.

4. Non-convergence and reported convergence. *(Example 4.5.14 pp 85-86 in Corbae et al. book)*. Let $f:\mathbb{R}_+\to\mathbb{R}_+,\ f(x)=x+\dfrac{1}{x+1}$. Consider the sequence $\left(x_n\right)$ constructed recursively by the rule $x_n=f\left(x_{n-1}\right)=x_{n-1}+\dfrac{1}{1+x_{n-1}}$, x_0 given, $n=1,2,\ldots$

Show a) that (x_n) does not converge (under the usual metric), but b) that any numerical procedure will eventually report convergence. Also, examine whether the sequence is Cauchy or not.

5. Convergence of product of function sequences. *(exercise 16.5 in Sutherland's "Introduction to Metric and Topological Spaces").* Suppose that $(f_n),(g_n)$ converge uniformly to f,g on D and that for each n f_n, g_n are bounded on D . Prove that $(f_n g_n)$ converges uniformly to fg on D.

6. Related to exercise 5, assume that we only know that only one of the two function sequences converges uniformly, while the other converges point-wise. Can we say that $(f_n g_n)$ converges? If yes, uniformly or point-wise? If not, provide a counter-example.

7. Related to exercise 5, assume that what we know is that both function sequences converge point-wise. Can we say that $(f_n g_n)$ converges point-wise? If not, provide a counter-example.

8. Composition of a function-sequence with a function. Assume that a sequence of functions (f_n) converges pointwise to some function f on D and let $g: H \to D$ be a function. Prove that the sequence $(f_n(g))$ converges pointwise to $f(g)$ on H.

9. Composition of a function with a function-sequence. Let f be a continuous function on D. Assume that a sequence of functions (g_n) converges pointwise to some function g on H and that all g_n and the limit g map H to D. Prove that $(f(g_n))$ converges to $f(g)$ on H.

10. Composition of two function-sequences. Assume that a sequence of functions (f_n) converges uniformly to some continuous function f on D and that a sequence of functions (g_n) converges pointwise to a function g on H. All g_n and the limit g map H to D. Prove that the sequence $(f_n(g_n))$ converges pointwise to $f(g)$ on H. Provide counterexamples, where some of the assumptions do not hold and consequently $(f_n(g_n))$ does not converge to $f(g)$ (it may converge somewhere else).

11. Metrics and completeness *(exercise 17.5 in Sutherland's "Introduction to Metric and Topological Spaces").* Consider the following metrics for $\mathbb{R}$

(a) $d(x,y)=|x^3-y^3|$ (b) $d(x,y)=|e^x-e^y|$ (c) $d(x,y)=|\tan^{-1}(x)-\tan^{-1}(y)|$.

For which of these metrics is the space $(\mathbb{R},d)$ complete?

12. Metrics and completeness *(example 10.6.3. from Sercoid's book).*

Consider the closed interval $X=[1,\infty]$. Endowed with the usual metric $d_u=|x-y|$ it is a complete metric space. Consider now the inverse metric $d_v(x,y)=|x^{-1}-y^{-1}|$. Show that the space (X,d_v) is not complete by finding a Cauchy sequence that does not converge in it.

13. Open subsets metrics and completeness. *(example 10.3.4. from Sercoid's book -an open subset of a complete metric space is not complete unless it is also closed, and we can achieve that by the appropriate metric).* Suppose that (X,d) is a complete metric space, and that U is a proper non-empty open subset of X. Define the distance function between a point and a set by $\operatorname{dist}_d(x,U)=\inf\{d(x,u)|u\in U\}$. Define the reciprocal function $f(x)=1/\operatorname{dist}_d(x,U)$. Show that under the metric $e(x,y)=d(x,y)+|f(x)-f(y)|$, the space (U,e) is complete. --