Athens University of Economics and Business
Department of Economics

Postgraduate Program - MSc in Economic Theory
Course: Mathematical Economics (Mathematics II)
Some Solutions to Exercises 2
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Exercise 1

Let (X, d) be a "pseudometric space" (i.e d is a pseudometric on $X \neq \varnothing$). For $x \in X$ and $\varepsilon > 0$ we can try and define d-open balls with center x and radius ε as

$$\mathcal{O}_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

and d-closed balls with center x and radius ε as

$$\mathcal{O}_d[x, \varepsilon] = \{y \in X : d(x, y) \leq \varepsilon\}$$

To argue that $\mathcal{O}_d(x, \varepsilon)$ and $\mathcal{O}_d[x, \varepsilon]$ can be defined for pseudometric spaces, it suffices to show that they are non-empty sets for any x and ε. Consider the following, for d pseudometric on X

$$\forall x \in X, \varepsilon > 0 \; \exists \; y \in X : d(x, y) = 0 < \varepsilon$$

So $\exists \; y \in X : y \in \mathcal{O}_d(x, \varepsilon)$, so $\mathcal{O}_d(x, \varepsilon) \neq \varnothing$. Analogously, $\mathcal{O}_d[x, \varepsilon] \neq \varnothing$. So open and closed balls can be defined for pseudometric spaces.

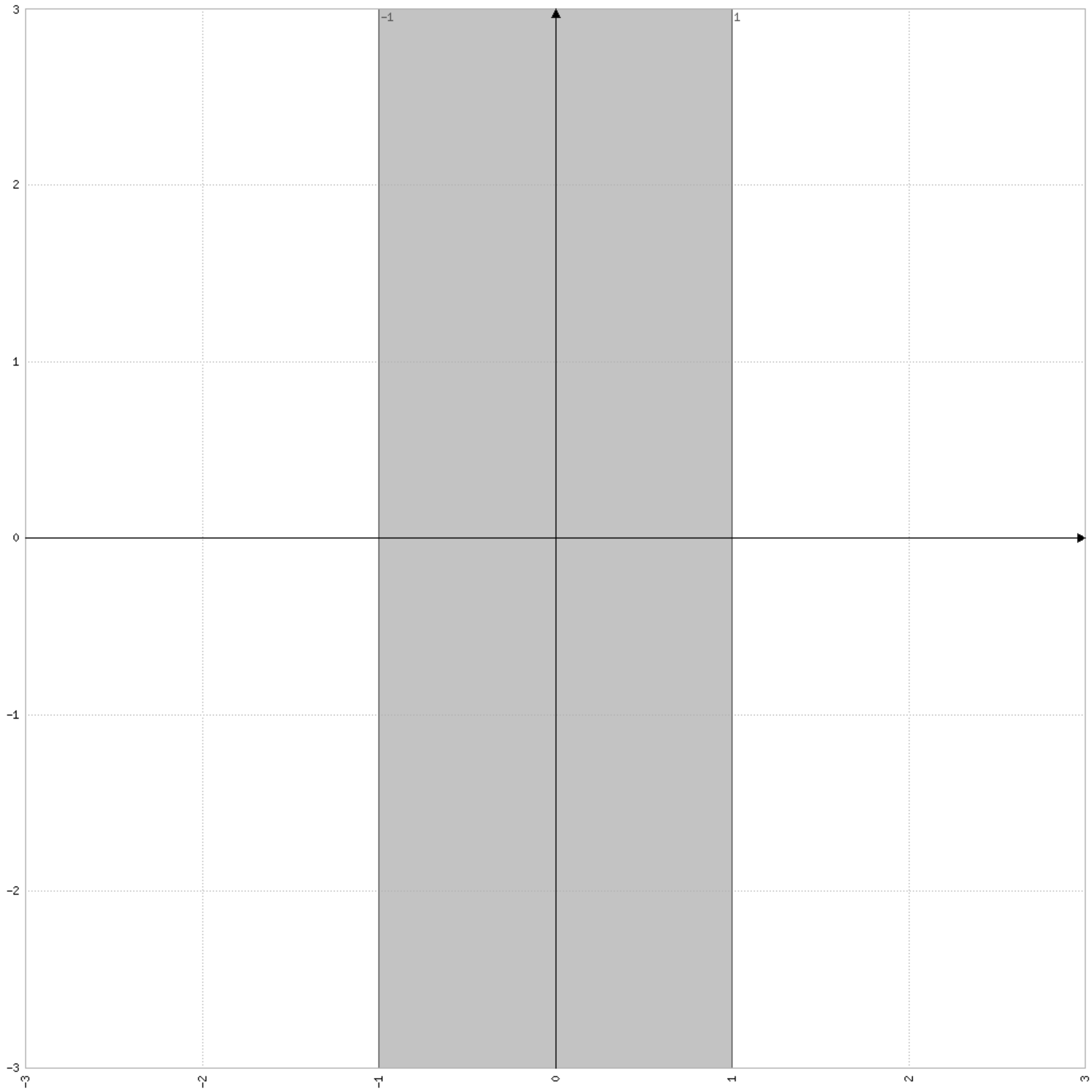
Exercise 2

$$
\begin{aligned}
d_A(x,y) &= \sqrt{(x-y)'A(x-y)} \\
&= \sqrt{\begin{bmatrix} x_1-y_1 & x_2-y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1-y_1 \\ x_2-y_2 \end{bmatrix}} \\
&= \sqrt{\begin{bmatrix} 1\cdot(x_1-y_1)+0\cdot(x_2-y_2) & 0\cdot(x_1-y_1)+0\cdot(x_2-y_2) \end{bmatrix} \begin{bmatrix} x_1-y_1 \\ x_2-y_2 \end{bmatrix}} \\
&= \sqrt{\begin{bmatrix} x_1-y_1 & 0 \end{bmatrix} \begin{bmatrix} x_1-y_1 \\ x_2-y_2 \end{bmatrix}} \\
&= \sqrt{(x_1-y_1)^2} \\
&= |x_1-y_1|
\end{aligned}
$$

It can be shown that d_A is a pseudometric on $\mathbb{R}^2$ (but not a metric). So we can define open balls on $(\mathbb{R}^2, d_A)$.

$$
\begin{aligned}
\mathcal{O}_{d_A}(\vec{0},1) &= \{y\in\mathbb{R}^2 : d_A(\vec{0},y)<1\} \\
&= \{y\in\mathbb{R}^2 : |0-y_1|<1\} \\
&= \{y\in\mathbb{R}^2 : |y_1|<1\} \\
&= \{y\in\mathbb{R}^2 : -1<y_1<1, y_2\in\mathbb{R}\} \\
&= (-1,1)\times\mathbb{R}
\end{aligned}
$$

And $\mathcal{O}_{d_A}(\vec{0},1)$ "looks like" the shaded area of the following graph **excluding** the boundaries. For $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the horizontal axis corresponds to x_1 and the vertical axis to x_2.



Exercise 4

Let $\varepsilon(x) \coloneqq \min_{y \in X, x \neq y} d(x, y)$ and $\delta \coloneqq \max_{x,y \in X} d(x, y)$. If X has a finite number of elements, then $\varepsilon(x), \forall x \in X$ and δ are positive and bounded real numbers. For each center $x \in X$ choose $\varepsilon(x)$ to be the radius of a d-open ball. Then $\mathcal{O}_d(x, \varepsilon(x)) = \{x\}$ for all $x \in X$. For some $x \in X$ choose δ as the radius of a d-open ball centred at x. Then $\mathcal{O}_d(x, \delta) = X$.

Exercise 6

Choose $x = 1 \in X$ and $0 < \varepsilon < 1$. Then $\forall y \in X : y \neq x$ it holds that $d(x, y) > 1 > \varepsilon$, while $d(1, 1) = 0 < \varepsilon$. So $\mathcal{O}_d(1, \varepsilon) = \{1\}$.

Exercise 7

$\mathcal{O}_{d^*}(x,\varepsilon) = \{y \in Y : d^*(x,y) < \varepsilon\}$

Although $d^* = d|_{Y\times Y} \neq d$ (because d^* and d have different domains), it holds numerically that $d^*(x,y) = d(x,y), \forall x,y \in Y \subseteq X$. So, for all $y \in X$ that belong to $\mathcal{O}_{d^*}(x,\varepsilon)$

$$\begin{aligned} & y \in \mathcal{O}_{d^*}(x,\varepsilon) \iff \\ & y \in \{y \in Y : d^*(x,y) < \varepsilon\} \iff \\ & y \in Y \text{ and } d^*(x,y) < \varepsilon \iff \\ & y \in Y \text{ and } d(x,y) < \varepsilon \iff \\ & y \in Y \text{ and } y \in \mathcal{O}_d(x,\varepsilon) \end{aligned}$$

so $\mathcal{O}_{d^*}(x,\varepsilon) = Y \cap \mathcal{O}_d(x,\varepsilon)$.

Similarly $\mathcal{O}_{d^*}[x,\varepsilon] = Y \cap \mathcal{O}_d[x,\varepsilon]$.

Exercise 10

(X,d) bounded space means that X is a d-bounded subset of itself, i.e. $\exists\, x \in X, \varepsilon > 0 : X \subseteq \mathcal{O}_d(x,\varepsilon)$, which means that for all $y \in X$ we have $d(x,y) < \varepsilon < +\infty$. So for all $f,g \in \mathcal{B}(Y,X)$

$$\begin{aligned} d^d_{sup}(f,g) &= \sup_{x\in Y} d(f(x),g(x)) \\ &< \sup_{x\in Y} \varepsilon \\ &= \varepsilon < +\infty \end{aligned}$$

So for any $f \in \mathcal{B}(Y,X)$ we have that $\forall g \in \mathcal{B}(Y,X) \Rightarrow d^d_{sup}(f,g) < \varepsilon$. By definition, this means that $\forall g \in \mathcal{B}(Y,X) \Rightarrow g \in \mathcal{O}_{d^d_{sup}}(f,\varepsilon)$ and thus $\mathcal{B}(Y,X) \subseteq \mathcal{O}_{d^d_{sup}}(f,\varepsilon)$. So $\mathcal{B}(Y,X)$ is d^d_{sup}-bounded and $(\mathcal{B}(Y,X), d^d_{sup})$ is bounded.