Athens University of Economics and Business
Department of Economics

Postgraduate Program - Master's in Economic Theory
Course: Mathematical Analysis (Mathematics II)
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Semester: Spring 2016-2017

18-03-2017

EXERCISES 2

Balls and Boundedness

1. Show that open and closed balls can be defined for pseudo-metric spaces.

2. Given the previous result, draw what $\mathrm{O}_{d_A}(0,1)$ looks like, for $X=\mathbb{R}^2$ and
$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

3. For $X=\mathbb{R}^3$ show that $d(x,y):=\max\left\{|x_2-y_2|,\sqrt{(x_1-y_1)^2+(x_3-y_3)^2}\right\}$ $x,y\in\mathbb{R}^3$ is a well-defined metric. Visualize $\mathrm{O}_d(\mathbf{0},1)$.

4. Suppose that X is finite. Show that every singleton $\{x\}, x\in X$, as well as X, is a d-open ball of (X,d).

5. Show that if (X,d) is a metric space, $y\in\mathrm{O}_d(x,\varepsilon)$, then
$\exists\delta>0:\ \mathrm{O}_d(y,\delta)\subseteq\mathrm{O}_d(x,\varepsilon)$

6. Let $X=(-\infty,0)\cup\{1\}$ and d the restriction of the usual metric (absolute value), in X. Show that $\exists x\in X,\varepsilon>0:\ \mathrm{O}_d(x,\varepsilon)=\{x\}$

7. More generally, show that if (Y,d^*) is a metric subspace of (X,d), i.e.
$Y\subseteq X,\ Y\neq\varnothing, d^*=d|_{Y\times Y}$, then $\forall x\in Y,\ \varepsilon>0$ we have

$\mathrm{O}_{d^*}(x,\varepsilon)=\mathrm{O}_d(x,\varepsilon)\cap Y$ and $\mathrm{O}_{d^*}[x,\varepsilon]=\mathrm{O}_d[x,\varepsilon]\cap Y$

8. Show that if d_1, d_2 are metrics with which X can be endowed, and $d_1 \le c d_2$ (as functions) for some $c \ge 0$ then $\forall x \in X,\ \varepsilon > 0\ \ \exists \delta > 0:\ \mathrm{O}_{d_2}(x,\delta) \subseteq \mathrm{O}_{d_1}(x,\varepsilon)$

9. For (X,d) a metric space, $Y \ne \varnothing$, define $B(Y,X)$ (the set of bounded functions from Y to X). Consider $d_{\sup}^{d} : B^2(Y,X) \to \mathbb{R},\quad d_{\sup}^{d}(f,y) = \sup_{x \in Y} d(f(x), y(x))$. Show that it is a well defined metric.

10. For the framework of the previous exercise, show that if (X,d) is bounded then $\left(B(Y,X), d_{\sup}^{d}\right)$ is bounded.

11. Show that if I is a finite index set, A_i are bounded subsets of (X_i, d_i), then $A := \prod_{i \in I} A_i$ is $d_{\Pi_{\mathrm{I}}}$ and $d_{\Pi_{||}}$ -bounded.

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