Athens University of Economics and Business
Department of Economics

Postgraduate Program - Master's in Economic Theory
Course: Mathematical Analysis (Mathematics II)
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EXERCISES 1

Metric Spaces and Distance Functions

1. Suppose that (X, d_x) is a metric space. For a set $Z \neq \varnothing$ suppose that $f: Z \to X$ is an injection. Prove that (Z, d_f) is a metric space, where

$d_f(x,y) = d_x(f(x), f(y)), \quad x, y \in Z$.

2. For (X,d) a metric space, show that $|d(x,z) - d(z,y)| \leq d(x,y) \quad \forall x,y,z \in X$

(***note***: Dual version of the triangle inequality).

3. Suppose that d is a metric on a set X . Prove that the inequality

$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$ holds for all $x,y,z,w \in X$.

(***hint*:** try to arrive at and use the result proved in the previous exercise. This is like a triangle inequality using four points rather than three).

4. Suppose (X,d) is a metric space, and $e(x,y) = d(x,y) / (1 + d(x,y))$ for each $x, y \in X$. Show that the function e is a metric on X .

5. "**The sum of two metrics is a metric**"**:** Assume a set X and two metrics on it, $d(x,y)$, $e(x,y)$ $x,y \in X$ and consider their sum, $v(x,y) = d(x,y) + e(x,y)$. Prove that $v(x,y)$ is a metric.

6. "**The sum of one metric and one pseudo-metric is a metric**"**:** Consider a set X , a metric $d(x,y)$ on it, and a pseudo-metric $e(x,y)$ on it, $x,y \in X$. Prove that $v(x,y) = d(x,y) + e(x,y)$ is a metric.

7. For (X,d) a metric space, suppose that X is also endowed with an additional operation, i.e. a function $+ : X \times X \to X$. Given this, d is termed as ***translation invariant*** iff

$\forall x,y,z \in Z \quad d(x,y) = d(x+z, y+z)$

Deduce which of the examples examined in class concern translation invariant metrics.

8. For $X = \mathbb{R}^k$, $k, p \geq 1$, prove that d_p defined by $d_p(x,y) = \left(\sum_{i=1}^{k} |x_i - y_i|^p\right)^{1/p}$ $x,y \in \mathbb{R}^k$, is a metric.

(***hint***: use Jensen's Inequality, along with the fact that $x \to x^p$ is convex for $p \geq 1$).

9. Extend the previous to the set $p-AS$ of real sequences where

$p-AS = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < +\infty \right\}$.

10. For $\alpha < \beta$, $p \geq 1$, consider $C([\alpha,\beta],\mathbb{R}) = \{ f : [\alpha,\beta] \to \mathbb{R}, continuous \}$. For

$X = C([\alpha,\beta],\mathbb{R})$ consider $d_p^*(f,g) = \left(\int_\alpha^\beta |f(x)-g(x)|^p \, dx\right)^{1/p}$. Prove that d_p^* is a metric.

11. Prove the following version of the Cauchy-Schwarz inequality: for $x, z \in \mathbb{R}^k$, $\mathbf{A}$ a $k \times k$ symmetric positive definite matrix, $x'\mathbf{A}z \leq \sqrt{x'\mathbf{A}x}\sqrt{z'\mathbf{A}z}$

12. Is the function $(x-y)'\,\mathbf{A}(x-y)$ a metric (squared Euclidean Distance) ?

13. Suppose I is a finite index set $I = \{1,2,\ldots,n\}$ and $\forall i \in I$ $(X_i, d_i) = (B(Y_i,\mathbb{R}),\, d_{\sup})$ is a metric space. Consider the product space $\prod_{i \in I} X_i$ and examine whether

$d_{\Pi,\,\mathrm{ms}} = \max_i \sup_{x \in Y_i} |f_i(x) - g_i(x)|$ is a metric in the product space.

14. State conditions under which the following statement is true: "The sum of two pseudo-metrics is a metric". Construct an example.

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