14/04/20 Lecture

(Continue the proof of that when A is t.b., the ball centers of the collections that cover it can be chosen to lie inside A, $\forall \varepsilon > 0$).

Reminder: For $\varepsilon > 0$, we considered $\mathcal{O}_{\varepsilon/2}(Z)$ a finite cover of open balls of radius $\varepsilon/2$ that covers A.

$Z = \{x_1, x_2, \dots, x_n\}$. We supposed that $x_1 \notin A$.

We considered the case $O_d(x_1, \varepsilon/2) \cap A$.

We have chosen $y \in O_d(x_1, \varepsilon/2) \cap A$, and considered the ball $O_d(y, \varepsilon)$: we have that $O_d(y, \varepsilon) \supseteq O_d(x_1, \varepsilon/2) \cap A$.

This is due to that: if $z \in O_d(x_1, \varepsilon/2) \cap A$, then

$$d(y,z) \leq \underbrace{d(x_1,y)}_{<\varepsilon/2} + \underbrace{d(x_1,z)}_{<\varepsilon/2} < \varepsilon/2 + \varepsilon/2 = \varepsilon \implies$$

z arbitrary

$$d(y,z) < \varepsilon \iff z \in O_d(y,\varepsilon) \implies O_d(y,\varepsilon) \supseteq O_d(x_1, \varepsilon/2) \cap A$$

Obviously we can repeat the previous procedure for any $x_i \in Z$ such that $x_i \notin A$.

Hence for any $x_i \in Z$ and $x_i \notin A$, and $O_d(x_i, \varepsilon/2) \cap A \neq \emptyset$, we can find some $y_i \in A$ such that $O_d(y_i, \varepsilon) \supseteq O_d(x_i, \varepsilon/2) \cap A$.

For each $x_i \in Z$, for which $x_i \in A$ we consider $O_d(x_i, \varepsilon)$.

Obviously due to monotonicity we have that

$O_d(x_i,\varepsilon) \supseteq O_d(x_i,\varepsilon/2) \cap A$. Hence we have constructed a finite collection of open balls of radius ε with:
i. centers inside A and ii. the collection covers A.
The result follows since ε is arbitrary. $\square$

Lemma. The previous hold if instead of covers by open balls we had considered covers of closed balls.

Proof. (covers of open balls $\Rightarrow$ covers of closed balls)
Suppose A is t.b. (based on covers of open balls) and $\varepsilon > 0$. There exists a cover of open balls, say,

$$\mathcal{O}_\varepsilon(Z) = \{ O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \dots, O_d(x_n,\varepsilon) \}$$

such that $\bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq A$. We have that $O_d(x_i,\varepsilon) \subseteq O_d[x_i,\varepsilon]$ $\forall i = 1,2,\dots,n \Rightarrow \bigcup_{i=1}^{n} O_d[x_i,\varepsilon] \supseteq A$. Hence

$\{ O_d[x_1,\varepsilon], O_d[x_2,\varepsilon], \dots, O_d[x_n,\varepsilon] \}$ is the required collection. (Hence the definition of t.b. using open balls implies the definition of t.b. using closed balls)

(covers of closed balls $\Rightarrow$ covers of open balls)

Suppose that A satisfies the definition of t.b. w.r.t. covers of closed balls. Let $\varepsilon>0$: we know that there exists a finite collection of closed balls

$$\{O_d[x_1,\varepsilon/2], O_d[x_2,\varepsilon/2], \dots, O_d[x_n,\varepsilon/2]\}$$ that covers A. We know that $O_d(x_i,\varepsilon) \supseteq O_d[x_i,\varepsilon/2]$ $\forall i=1,\dots,n$

$$\Rightarrow \bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq \bigcup_{i=1}^{n} O_d[x_i,\varepsilon/2] \supseteq A$$ hence the collection $\{O_d(x_1,\varepsilon), O_d(x_2,\varepsilon),\dots, O_d(x_n,\varepsilon)\}$ is the required one. Since ε is arbitrary the result follows. $\square$

Further useful results:

- X,d will be considered totally bounded iff X is totally bounded w.r.t. as a subset of itself.
- $A\subseteq X$ is totally bounded w.r.t. d, iff A, d_A is totally bounded (Proof - Exercise! Use the result on the choice of the ball centers - it is similar to the proof of the analogous result for boundness).

Lemma If A is totally bounded w.r.t. d then it is also bounded w.r.t. d.

Proof. [check the definitions using the covers!]

Lemma. If A is finite then it is totally bounded. (this holds for any metric!)

Proof. If $A = \emptyset$ then A is trivially t.b. Suppose that $A \neq \emptyset$, i.e. it is of the form

$$A = \{ x_1, x_2, \dots, x_n \}, \quad x_i \in X \ \forall i = 1, \dots, n$$

For $\varepsilon > 0$, consider $\{ O_d(x_1, \varepsilon), O_d(x_2, \varepsilon), \dots, O_d(x_n, \varepsilon) \}$

Since $x_i \in O_d(x_i, \varepsilon) \ \forall i = 1, \dots, n$, we have that

$$\bigcup_{i=1}^{n} O_d(x_i, \varepsilon) \supseteq A.$$ Since ε is arbitrary the result follows. $\square$

Lemma. If A is t.b. w.r.t. d and $B \subseteq A$ then B is also t.b. w.r.t. d.

Proof. $\forall \varepsilon > 0$ the cover of open balls covers A it necessarily covers B.

Dually: if A is not t.b. w.r.t. d and $B \supseteq A$ then also B is not t.b. w.r.t. d

(A is not t.b. iff $\exists \varepsilon>0$: every finite collection of open balls of radius ε cannot cover A).

Examples and Counter-examples

1. X, d_d (X, d_d is always bounded)

We will show that: X, d_d is totally bounded iff it is finite.

Proof a. (Finite $\Rightarrow$ t.b) (see above).

b. (if X, d_d is t.b. then it is finite).

Suppose that X is infinite. Let $\varepsilon = 1/2$.
(Reminder $O_{d_d}(x, 1/2) = \{x\}$ $\forall x \in X$). This directly implies that the only way to cover X by open balls of radius $1/2$ is: $\bigcup_{x\in X} O_{d_d}(x, 1/2)$

$$\bigcup_{x\in X} O_{d_d}(x,1/2) = \bigcup_{x\in X} \{x\} = X$$

but the collection $\{O_{d_d}(x, 1/2), x \in X\}$ is not finite since X is infinite. The result follows. $\square$

Hence the previous implies via example that total boundness is strictly stronger than boundness.

2. Is $\mathbb{R}, d_u$ totally bounded? No since it is not even bounded. The same holds for $\mathbb{R}^d, d_I$.

(the interesting question here is when $A \subseteq \mathbb{R}^d$ is t.b. w.r.t. d_I).

3. Remember $B(\mathbb{N}, \mathbb{R}), d_{sup}$ (bounded real sequences with the uniform metric).

$$A = \left\{ f_M \in B(\mathbb{N}, \mathbb{R}),\ f_M(n) = \begin{cases} 1 & M = n \\ 0 & M \neq n \end{cases},\ M \in \mathbb{N} \right\}$$

Reminder: infinite dimensional vector representation of f_M

$$(0, 0, \ldots, 1, 0, \ldots 0, \ldots)$$

$(M+1)^{th}$ position

e.g. f_0 $\quad (1, 0, 0, \ldots, 0, \ldots)$

We have proven that A is bounded w.r.t.

d_{sup} (we have used uniform boundness).

Is A totally bounded w.r.t. d_{sup}? A is Not totally bounded due to that: we know that if A were t.b. then the centers of the Covering balls can be chosen to belong to A. Also we have that $d_{sup}(f_n, f_m) = \begin{cases} 0, & n=m \\ 1, & n\neq m. \end{cases}$ A is obviously infinite. For $\varepsilon = 1/2$

$$O_{d_{sup}}(f_m, 1/2) \cap A = \{f_m\}.$$

Hence in order to be able to cover A with open balls of the form $O_{d_{sup}}(f_m, 1/2)$, $m \in \mathbb{N}$ we need as many of those balls as there are elements of. But A is infinite. Hence A is not totally bounded w.r.t. d_{sup}. □

In order to construct examples the following Notion will be usefull:

Covering Numbers and Metric Entropy

We will use without loss of generality covers by closed balls.

Framework: X, d, $A \subseteq X$, $\varepsilon > 0$

Definition: The covering number of A w.r.t. ε and d, $N(\varepsilon, d, A) :=$ Min # of closed balls of radius ε that cover A.

Remark: it is possible to prove that $N(\varepsilon, d, A)$ is unique $\forall \varepsilon, d, A$.

$\ln N(\varepsilon, d, A)$ is the Metric entropy number w.r.t. ε and d.

- It is easy to show that A is t.b. w.r.t. d iff $N(\varepsilon, d, A) \in \mathbb{N}$ (i.e. it is $< +\infty$) $\forall \varepsilon > 0$.

(dually A is not t.b. w.r.t. d iff $\exists \varepsilon > 0$: $N(\varepsilon, d, A) = +\infty$)

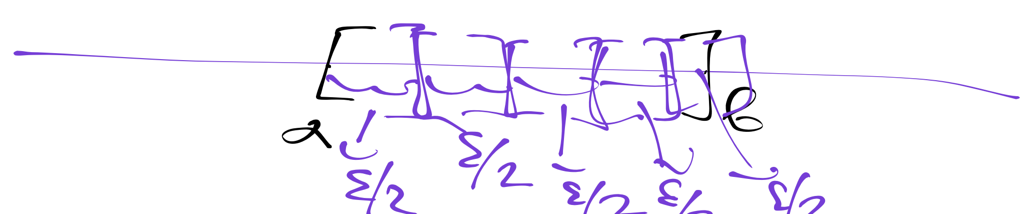
- It is easy to see that $N(\varepsilon, d, A)$ is decreasing w.r.t. ε for fixed d, A.

- Generally, as $\varepsilon \downarrow 0$ $\left(\begin{matrix}\varepsilon \to 0 \\ \varepsilon > 0\end{matrix}\right)$, $N(\varepsilon, d, A) \to +\infty$ for fixed d, A even when A is t.b.

- What is very informative on the complexity of A as a metric (sub-)space is the rate at which $N(\varepsilon, d, A)$ (or equivalently $\ln N(\varepsilon, d, A)$) diverges as $\varepsilon \downarrow 0$.

E.g. $X = \mathbb{R}$, $d = d_u$

$$[\alpha, \beta] = O_{d_u}\left[\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}\right]$$



$$N(\varepsilon, d_u, [\alpha, \beta]) = \begin{cases} 1, & \varepsilon \geq \frac{\beta-\alpha}{2} \\ \text{ο μικρότερος φυσικός} \geq \frac{\beta-\alpha}{2\varepsilon}, & \varepsilon < \frac{\beta-\alpha}{2} \end{cases} < +\infty$$

$\forall \varepsilon > 0$. Hence $[\alpha, \beta]$ is t.b. w.r.t. d_u

Hence we have shown that $\forall x \in \mathbb{R}, \varepsilon > 0$, $O_{d_u}[x, \varepsilon]$ is totally bounded w.r.t. d_u. Hence if $A \subseteq \mathbb{R}$ is bounded w.r.t. d_u then it is also t.b. w.r.t. d_u.

Hence Boundness $\Leftrightarrow$ Total Boundness inside $\mathbb{R}, d_u$