Lecture 28/05/20

Applications of BFPT: Picard-Lindelöf Theorem

Reminder: - $Z = [\alpha, \beta]$ $d_Z = d_u$, compactness

- $H: Z \times \mathbb{R} \to \mathbb{R}$, jointly continuous, $\exists \delta > 0$:

$\forall x \in Z, y_1, y_2 \in \mathbb{R}$

$$|H(\underset{=}{\overset{z}{x}}, \overset{f(z)}{y_1}) - H(\overset{z}{x}, \overset{g(z)}{y_2})| \leq \delta |y_1 - y_2| \quad \checkmark$$

[Lipschitz continuity of $H(x,y)$ w.r.t. y uniformly in x.]

- $x_0 \in Z$, $y_0 \in \mathbb{R}$

$*$ BVP:
$$\begin{cases} f'(x) = H(x, f(x)), & \forall x \in [\alpha, \beta] \quad \checkmark \\ f(x_0) = y_0 & \checkmark \end{cases}$$

P-L Theorem: BVP has a unique solution in $C([\alpha,\beta], \mathbb{R})$

Proof: (We will use our BFPT methodology. At first it would be helpfull if we obtained an equivalent formulation of BVP that includes only f instead of f' - this way we could hope to identify some Φ the fixed points of which match the solutions of BVP.)

We thus integrate $(*)$. Then:

$$(*) \Leftrightarrow \begin{cases} \displaystyle\int_{x_0}^{x} f'(z)\,dz = \underline{\int_{x_0}^{x} H(z,f(z))\,dz} & \forall x \in [\alpha,\beta] \\ f(x_0) = y_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} f(x) - f(x_0) = \displaystyle\int_{x_0}^{x} H(z,f(z))\,dz & \forall x \in [\alpha,\beta] \\ f(x_0) = y_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} f(x) = f(x_0) + \displaystyle\int_{x_0}^{x} H(z,f(z))\,dz & \forall x \in [\alpha,\beta] \\ f(x_0) = y_0 \end{cases}$$

$$\Leftrightarrow \quad f(x) = y_0 + \int_{x_0}^{x} H(z,f(z))\,dz \quad \forall x \in [\alpha,\beta] \quad (**)$$

Remark: the integral equation $(**)$ is equivalent to the BVP (BVP and $(**)$ have the same solutions]. It suffices to prove that $(**)$ has a unique solution in $C([\alpha,\beta],\mathbb{R})$. $(**)$ contains f and not f' in both sides.

Remark: Consider $y_0 + \int_{x_0}^{x} H(z,f(z))\,dz$. Due to the continuity of H and f, $\int_{x_0}^{x} H(z,f(z)) \in \mathbb{R}$ $\forall x \in [\alpha,\beta]$, $\forall f \in C([\alpha,\beta],\mathbb{R})$. Hence $y_0 + \int_{x_0}^{x} H(z,f(z))\,dz$ is a well-defined function $[\alpha,\beta] \to \mathbb{R}$, $\forall f \in C([\alpha,\beta],\mathbb{R})$. $\forall x_n, x \in [\alpha,\beta]$, $x_n \to x$, $\int_{x_0}^{x_n} H(z,f(z))\,dz \to \int_{x_0}^{x} H(z,f(z))\,dz$, hence $y_0 + \int_{x_0}^{x} H(z,f(z))\,dz \in C([\alpha,\beta],\mathbb{R})$.

Define Φ on $C([a,b],\mathbb{R})$ as follows:

if $f \in C(C[a,b],\mathbb{R})$, $(\Phi(f))(x) := y_0 + \int_{x_0}^{x} H(z,f(z))dz$, $\forall x \in [a,b]$.

Due to the previous remark $\Phi(f)$ is a well-defined function $[\alpha,b] \to \mathbb{R}$ that is also continuous. Hence Φ is a self map on $C(C[\alpha,b],\mathbb{R})$ (i.e. $\Phi: C([\alpha,b],\mathbb{R}) \to C([a,b],\mathbb{R})$).

Furthermore f is a fixed point of $\Phi \iff$

$$f = \Phi(f) \iff$$

$$f(x) = (\Phi(f))(x) \quad \forall x \in [a,b] \iff$$

$$f(x) = y_0 + \int_{x_0}^{x} H(z,f(z))dz \, , \ \forall x \in [a,b] \iff$$

f is a solution of $(**)$.

Hence in order to prove P-L Theorem it suffices to prove that Φ has a unique fixed point. We will use BFPT:

We have that:

– $X = C([\alpha,b],\mathbb{R}) \underset{\substack{[a,b] \text{ is} \\ \text{Compact}}}{\subseteq} B(C[a,b],\mathbb{R})$

– We know from previous remarks (ii) that $C(C[a,b],\mathbb{R}), d_{sup}$ is complete. It is possible to show (exercise!!!) that Φ is a d_{sup}-contraction under further restrictions on x_0, δ, α, b which do not appear in the P-L Theorem. Such restrictions would constitute a "local" version of the theorem.

We will work with d^*_{sup} defined as: $f,g \in C([a,b],\mathbb{R})$

$$d^*_{sup}(f,g) = \sup_{x\in[a,b]}\left(e^{-\underbrace{\delta}_{\text{depends on BVP}}(x-x_0)}|f(x)-g(x)|\right)$$

$\neq 0 \ \forall x\in[a,b]$

d^*_{sup} is a "modification" of d_{sup} in a way that "reflects" properties of the BVP.

Exercise: Show that d^*_{sup} is a well-defined metric.

We thus endow $C([a,b],\mathbb{R})$ with d^*_{sup}.

1. Completeness: $e^{-\delta(b-x_0)}|f(x)-g(x)| \leq e^{-\delta(x-x_0)}|f(x)-g(x)|$

$$\leq e^{-\delta(a-x_0)}|f(x)-g(x)| \quad \forall x\in[a,b]$$

$$\Rightarrow \sup_{x\in[a,b]} \overbrace{e^{-\delta(b-x_0)}}^{>0}|f(x)-g(x)| \leq \sup_{x\in[a,b]} e^{-\delta(x-x_0)}|f(x)-g(x)|$$

$$\leq \sup_{x\in[a,b]} \overbrace{e^{-\delta(a-x_0)}}^{>0}|f(x)-g(x)| \quad \Leftrightarrow$$

$$e^{-\delta(b-x_0)}\underbrace{\sup_{x\in[a,b]}|f(x)-g(x)|}_{d_{sup}(f,g)} \leq d^*_{sup}(f,g) \leq e^{-\delta(a-x_0)}\sup_{x\in[a,b]}|f(x)-g(x)|$$

$$\Leftrightarrow e^{-\delta(b-x_0)}d_{sup}(f,g) \leq d^*_{sup}(f,g) \leq e^{-\delta(a-x_0)}d_{sup}(f,g)$$

Hence we have: $c_1 d_{sup} \leq \underline{d^*_{sup}} \leq c_2 d_{sup}$, $c_1 = e^{-\delta(b-x_0)}$, $c_2 = e^{-\delta(a-x_0)}$

Then we know $C(C[\alpha,\beta],\mathbb{R}), d_{sup}$ is complete $\Longrightarrow$ due to our results on completeness

$C(C[\alpha,\beta],\mathbb{R}), d^*_{sup}$ is complete due to the previous inequalities.

(This essentially justifies all our previous work on Metric comparisons).

2. Contractivity: we will use the definition, we cannot use Blackwell's Lemma since we are working with d^*_{sup}.

Let $f,g \in C(C[\alpha,\beta],\mathbb{R})$, we have that

$$d^*_{sup}(\underline{\Phi(f),\Phi(g)}) = \sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\left|(\Phi(f))(x)-(\Phi(g))(x)\right|\right]$$

$$=\sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\left|\cancel{y_0}+\int_{x_0}^{x}H(z,f(z))dz-\left(\cancel{y_0}+\int_{x_0}^{x}H(z,g(z))\,dz\right)\right|\right]$$

$$=\sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\left|\int_{x_0}^{x}\left(H(z,f(z))-H(z,g(z))\right)dz\right|\right]$$

$$\leq \sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\int_{x_0}^{x}\left|H(z,f(z))-H(z,g(z))\right|dz\right]$$

Lipschitz property + integral Monotonicity

$$\leq \sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\int_{x_0}^{x}\delta\,\underline{\left|f(z)-g(z)\right|}\,dz\right]$$

$$= \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} \delta\, e^{-\delta(z-x_0)} e^{\delta(z-x_0)} \left| f(z)-g(z) \right| dz \right]$$

take sup w.r.t. z

$$= \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} e^{-\delta(z-x_0)} \left| f(z)-g(z) \right| \delta e^{\delta(z-x_0)} dz \right]$$

Monotonicity of integral

$$\leq \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} \underbrace{\left(\sup_{z\in[\alpha,\beta]} \left(e^{-\delta(z-x_0)} \left| f(z)-g(z) \right| \right) \right)}_{\to\, d^*_{sup}(f,g)} \delta e^{\delta(z-x_0)} dz \right]$$

$$= \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} d^*_{sup}(f,g)\, \delta e^{\delta(z-x_0)} dz \right]$$

$d^*_{sup}(f,g) \geq 0$ and independent of z, x

$$= d^*_{sup}(f,g) \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} \delta e^{\delta(z-x_0)} dz \right]$$

$$\left(e^{\delta(z-x_0)} \right)' = \delta e^{\delta(z-x_0)}$$

$$= d^*_{sup}(f,g) \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \, e^{\delta(z-x_0)} \Big|_{x_0}^{x} \right] =$$

$$= d^*_{sup}(f,g) \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \left(e^{\delta(x-x_0)} - 1 \right) \right]$$

$$= d^*_{sup}(f,g) \sup_{x\in[\alpha,\beta]} \left(1 - e^{-\delta(x-x_0)} \right) = \left(1 - e^{-\delta(\beta-x_0)} \right) d^*_{sup}(f,g)$$

We have shown that: $\forall f,g \in C([a,b],\mathbb{R})$

$$d^*_{sup}(\Phi(f),\Phi(g)) \leq \left(1-e^{-\delta(b-x_0)}\right) d^*_{sup}(f,g)$$

We have that $0 < e^{-\delta(b-x_0)} \leq 1 \Leftrightarrow 0 \leq 1-e^{-\delta(b-x_0)} < 1$

Hence the d^*_{sup} - Lipschitz coefficient of Φ is < 1

and Φ is a d^*_{sup} - contraction.

Hence due to the BFPT Φ has a unique fixed point $\Leftrightarrow$ (**) has a unique solution $\Leftrightarrow$ BVP has a unique solution. $\square$