Lecture 26/05/20

Reminder:

* BFPT: If X admits a d, such that i. (X,d) is complete, and ii. $f: X \to X$ is a d-contraction, then f has a unique fixed point in X, say x^*, and $\forall y \in X$, $x^* = \lim_{n\to\infty} x_n$ (w.r.t. d), where $x_n = f^{(n)}(y)$.

Corollary [Further locating x^*]. If A is a closed and non empty subset of X, and $\forall x \in A,\ f(x) \in A$, then $x^* \in A$.

↳ f remains a self map when restricted to A.

Proof. X,d is complete and A is closed (w.r.t. d) subset of X. Hence (A,d) is a complete metric subspace of X. When f is restricted to A it remains a self-map. $f: X \to X$ is a d-contraction, hence it is also a d-contraction when restricted to A. Hence due to BFPT f restricted to A has a unique fixed point inside A. But we also know that f has a unique fixed point, x^*, in $X \supseteq A$. Hence $x^* \in A$. □

Corollary. If A is closed and $x^* \notin A$ then $\exists x \in A: f(x) \notin A$.

Applications of BFPT:

General Framework:

- X will be generally a subset of $B(Z,\mathbb{R})$ for Z some set and d will be either d_{sup} or some "modification" of d_{sup}.
- Φ will denote the respective self-map on X. (i.e. if f lies in the aforementioned function space, $\Phi(f)$ will also lie there).
- We will study problems of existence and uniqueness of solutions of equations on such function spaces (functional equations)
- The crucial step will be to represent the issue of existence and uniqueness of solutions by the issue of the existence and uniqueness of fixed points of a suitable Φ. In this respect we will have to:
  * identify Φ as the self-map the fixed points of which are the solutions of the equations
  * show that Φ is well-defined (i.e. that it is a self-map)
  * ensure that BFPT is applicable in this framework.

- One possible difficulty is to show that Φ is a contraction. One way to show contractivity by avoiding the definition goes through Blackwell's lemma. This runs as follows:

- $d = d_{sup}$
- $X \subseteq B(Z, \mathbb{R})$ contains every constant function.

Furthermore, if $f \in X$, $\alpha \in \mathbb{R}$ then $f + \alpha \in X$.

(α can be perceived as a constant function $Z \to \mathbb{R}$, $\alpha(x) = \alpha$ $\forall x \in Z$)

$(f+\alpha)(x) = f(x) + \alpha(x) = f(x) + \alpha$

(e.g. $B(Z, \mathbb{R})$ satisfies the above)

- if $f, g \in X$ then $f \gtrsim g \Leftrightarrow f(x) \geq g(x)$ $\forall x \in Z$.

(partial order on X)

Blackwell Hypotheses:

BL1. if $f \gtrsim g \Rightarrow \phi(f) \gtrsim \phi(g)$ (Monotonicity of ϕ)

$\Updownarrow$

$\left[\begin{array}{l} \phi(f), \phi(g) : Z \to \mathbb{R} \in X \\ (\phi(f))(x) \geq (\phi(g))(x), \ \forall x \in Z \end{array} \right]$

BL2. $\exists\, \delta \in (0,1)$: $\forall f \in X, \alpha \in \mathbb{R}$

$$\phi(f) + \delta\alpha \gtrsim \phi(f + \alpha)$$

$\phi(f)$: $Z \to \mathbb{R}$; $\delta\alpha$: constant function $Z \to \mathbb{R}$

Hence $\phi(f) + \delta\alpha \in X$

Blackwell's Lemma: Under BL1 and BL2 ϕ is a d_{sup}-contraction.

Proof. Let $f, g \in X$. $f(x) - g(x) \leq |f(x) - g(x)| \leq \sup_{x \in Z} |f(x) - g(x)| = d_{sup}(f, g)$ $\forall x \in Z$

$$\Longrightarrow \quad f(x) \le g(x) + d_{sup}(f,g) \quad \forall x \in Z \quad (*)$$

Define $g^*: Z \to \mathbb{R}$, by $g^*(x) := g(x) + d_{sup}(f,g)$, $\forall x \in Z$

then $g^* \in X$.

Due to $(*)$ $\quad g^* \gtrsim f$.

Analogously we can show that:

$$g(x) \le f(x) + d_{sup}(f,g) \quad \forall x \in Z$$

Define $f^*: Z \to \mathbb{R}$, by $f^*(x) := f(x) + d_{sup}(f,g)$, $\forall x \in Z$

and $f^* \in X$.

Analogously $\quad f^* \gtrsim g$.

Hence by the above constructions we have:

$$\left\{\begin{array}{l} g^* \gtrsim f \\ f^* \gtrsim g \end{array}\right. \xRightarrow[\Phi \text{ is Monotone by BL1}]{f,g,f^*,g^* \in X} \left\{\begin{array}{l} \Phi(g^*) \gtrsim \Phi(f) \\ \Phi(f^*) \gtrsim \Phi(g) \end{array}\right. \quad (*)$$

$\in X$

$$\Phi(g^*) = \Phi(g + d_{sup}(f,g)) \underset{BL2}{\lesssim} \Phi(g) + \delta\, d_{sup}(f,g)$$

$$\Phi(f^*) = \Phi(f + d_{sup}(f,g)) \lesssim \Phi(f) + \delta\, d_{sup}(f,g) \quad (**)$$

$$\left.\begin{array}{l} \Phi(g) + \delta\, d_{sup}(f,g) \gtrsim \Phi(g^*) \gtrsim \Phi(f) \\ \Phi(f) + \delta\, d_{sup}(f,g) \gtrsim \Phi(f^*) \gtrsim \Phi(g) \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l}\Phi(g)+\delta\, d_{sup}(f,g) \geq \Phi(f) \\ \Phi(f)+\delta\, d_{sup}(f,g) \geq \Phi(g)\end{array}\right\} \iff$$

$$\left.\begin{array}{l}\big(\Phi(g)\big)(x)+\delta\, d_{sup}(f,g) \geq \big(\Phi(f)\big)(x) \\ \big(\Phi(f)\big)(x)+\delta\, d_{sup}(f,g) \geq \big(\Phi(g)\big)(x)\end{array}\right\} \forall x \in Z \iff$$

$$\left.\begin{array}{l}\big(\Phi(f)\big)(x)-\big(\Phi(g)\big)(x) \leq \delta\, d_{sup}(f,g) \\ \big(\Phi(g)\big)(x)-\big(\Phi(f)\big)(x) \leq \delta\, d_{sup}(f,g)\end{array}\right\} \forall x \in Z \implies$$

$$\left|\big(\Phi(f)\big)(x)-\big(\Phi(g)\big)(x)\right| \leq \delta\, d_{sup}(f,g) \quad \forall x \in Z$$

and since $\delta\, d_{sup}(f,g)$ is independent of x, we have that

$$\sup_{x\in Z}\left|\big(\Phi(f)\big)(x)-\big(\Phi(g)\big)(x)\right| \leq \delta\, d_{sup}(f,g) \iff$$

$$d_{sup}\big(\Phi(f),\Phi(g)\big) \leq \delta\, d_{sup}(f,g)$$

Since $\delta\in(0,1)$ and f,g are arbitrary the previous shows that Φ is a d_{sup}-contraction as requested. $\square$

Further Structure:

i. We will assume that Z is endowed with the metric d_Z, w.r.t. which it is totally bounded and complete (Z is then termed compact w.r.t. the topology generated by d_Z).

ii. It is possible to prove that if Z, d_Z is compact then $C(Z,\mathbb{R}) := \{ f: Z \to \mathbb{R},\ f\ d_u/d_Z\text{-continuous} \}$ is a closed subset of $B(Z,\mathbb{R})$ w.r.t. d_{sup}. Since $B(Z,\mathbb{R}), d_{sup}$ is complete, this implies that $C(Z,\mathbb{R}), d_{sup}$ is a complete metric subspace.

iii. If Z, d_Z is compact and $f \in C(Z,\mathbb{R})$ then $\operatorname{argmax}_{x \in Z} f(x) \neq \emptyset$.

Application: Bellman Equation

Z, d_Z is compact. $\omega: Z \times Z \to \mathbb{R}$ is jointly continuous, $\delta \in (0,1)$.

Definition: [Bellman Equation]

$$(*) \qquad f(x) = \sup_{\underline{\underline{y \in Z}}} \left[\omega(x,y) + \underline{\underline{\delta}}\, f(y) \right], \quad f \in C(Z,\mathbb{R}).$$

Lemma. Bellman equation has a unique solution inside $C(Z,\mathbb{R})$.

Remark: $\omega(x,y)+\delta f(y)$ is jointly continuous due to the continuity of ω, f. For any x, $\omega(x,y)+\delta f(y)$ is then continuous w.r.t. y. $y \in Z$, and Z is compact hence due to iii

$\underset{y\in Z}{\operatorname{argmax}}[\omega(x,y)+\delta f(y)] \neq \emptyset \ \forall x \in X$ and thereby not only the

Supremum in the equation exists and $\sup_{y \in Z}[\omega(x,y)+\delta f(y)] =$

$\max_{y\in Z}[\omega(x,y)+\delta f(y)] \ \forall x \in Z$. Since sup in this context is

Continuous (Remember our first application!) $\max_{y \in Z}[\omega(x,y)+\delta f(y)]$ is a continuous function of x, i.e. $\max_{y\in Z}[\omega(x,y)+\delta f(y)] \in C(Z,\mathbb{R})$.

$\square$

Proof of Lemma.

$X = C(Z,\mathbb{R})$ and $d = d_{sup}$. (this makes sense due to ii).

Consider Φ defined on $C(Z,\mathbb{R})$ as:

$$(\Phi(f))(x) := \max_{y\in Z}[\omega(x,y)+\delta f(y)], \quad f \in C(Z,\mathbb{R}), \ x \in Z.$$

Due to the previous remark Φ is well-defined as a function $C(Z,\mathbb{R}) \to C(Z,\mathbb{R})$ (i.e. it is a self map on $C(Z,\mathbb{R})$).

Furthermore
$$(*) \iff f(x) = \max_{y\in Z}[\omega(x,y)+\delta f(y)] \ \forall x \in Z$$
$$\iff f(x) = (\Phi(f))(x) \ \forall x \in Z$$

$$\Leftrightarrow f = \Phi(f)$$

Hence f satisfies Bellman Equation iff it is a fixed point of Φ.

Hence in order to prove the Lemma it suffices to prove that Φ has a unique fixed point. We will use the BFPT.

α. We have that $(C(Z,\mathbb{R}), d_{sup})$ is complete due to ii above.

b. we will use Blackwell's Lemma in order to show that Φ is a d_{sup}-contraction:

I. constant functions $Z \to \mathbb{R}$ are obviously continuous and and if f continuous and α constant then $f+\alpha$ continuous $Z \to \mathbb{R}$.

II. BLL: Let $f,g \in C(Z,\mathbb{R})$ and $f \gtrsim g \Leftrightarrow$

$$f(y) \geq g(y) \ \forall y \in Z \underset{\delta>0}{\Leftrightarrow} \delta f(y) \geq \delta g(y) \ \forall y \in Z$$

$$\Rightarrow w(x,y) + \delta f(y) \geq w(x,y) + \delta g(y) \quad \forall x,y \in Z$$

$$\Rightarrow \sup_{y\in Z}[w(x,y)+\delta f(y)] \geq \sup_{y \in Z}[w(x,y)+\delta g(y)] \ \forall x \in Z$$

Monotonicity of sup

$$\Rightarrow (\Phi(f))(x) \geq (\Phi(g))(x) \ \forall x \in Z \Leftrightarrow \Phi(f) \gtrsim \Phi(g).$$

Since f, g are arbitrary, Φ is Monotone and BLL holds.

III. BL2. For the δ that appears in the equation, if $f \in C(Z, \mathbb{R})$, $\alpha \in \mathbb{R}$ we have

$$\Big(\Phi(f+\alpha)\Big)(x) = \sup_{y \in Z}\left[w(x,y) + \delta(f(y)+\alpha)\right] \quad \forall x \in Z$$

$$= \sup_{y \in Z}\left[w(x,y) + \delta f(y) + \underline{\delta\alpha}\right] \quad \forall x \in Z$$

$$= \sup_{y \in Z}\left[w(x,y) + \delta f(y)\right] + \delta\alpha \quad \forall x \in Z$$

$\delta\alpha$ is independent of y

$$= \Big(\Phi(f)\Big)(x) + \delta\alpha \quad \forall x \in Z$$

Hence we have shown that: $\Big(\Phi(f+\alpha)\Big)(x) = \Big(\Phi(f)\Big)(x) + \delta\alpha \;\; \forall x \in Z$

$$\Leftrightarrow \Phi(f+\alpha) = \Phi(f) + \delta\alpha$$

Hence BL2 holds as equality.

Hence Blackwell's lemma applies and thereby Φ is a $d_{\sup}$ contraction.

Hence BFPT applies and thereby Φ has a unique fixed point, and thereby Bellman equation has a unique solution. $\square$

Application: Picard - Lindelöf Theorem

- $Z = [\alpha, \beta]$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$. Z is compact w.r.t. d_a.

- $H: [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $\exists\ \delta > 0$: $\forall x \in [\alpha, \beta]$, $\forall y_1, y_2 \in \mathbb{R}$ we have:

$$\left| H(x, y_1) - H(x, y_2) \right| \leq \delta \, |y_1 - y_2|$$

($H(x, y)$ is Lipschitz continuous w.r.t. y, $\forall x \in X$ and the Lipschitz coefficient δ, does not depend on x.)

- $x_0 \in [\alpha, \beta]$, $y_0 \in \mathbb{R}$.

Consider the following Boundary Value Problem (BVP):

$$(*) \quad \begin{cases} f'(x) = H(x, \underline{\underline{f(x)}}), & \forall x \in [\alpha, \beta] \\ f(x_0) = y_0 \end{cases}$$

Picard - Lindelöf Theorem. There exists a unique $f \in C([\alpha, \beta], \mathbb{R})$ that solves $(*)$.

- $X = C\big([\alpha, \beta], \mathbb{R}\big)$

- to be continued...