Lecture 21/05/20

We are ready for the Banach Fixed Point Theorem (BFPT)

* X is a non-empty set. A self-map on X is any function $X \xrightarrow{f} X$. (such a function transforms X to itself)

* f is a self-map on X. Define: $(n \in \mathbb{N})$

$$f^{(n)} := \begin{cases} id_X, & n = 0 \\ \underbrace{f \circ f \circ \ldots \circ f}_{n\text{-fold}}, & n > 0 \end{cases} \qquad id_X : X \to X,\ id_X(y) = y \ \forall y \in X$$

$f^{(n)}$ is a self-map on X.

* Remember that composition is associative: $f^{(n)} = f^{(n_1)} \circ f^{(n_2)}$ whenever $n_1 + n_2 = n$.

* Suppose that X is endowed d, (X,d) is a m.s. Let f be a self map on X. X as both the domain and the range of f is considered endowed with the SAME metric d.

* Given the above: a self map on X is a $(d\text{-})$ contraction iff f is $(d/d\text{-})$ Lipschitz continuous and $C_f < 1$, i.e.

$$\forall x, y \in X \qquad d(f(x), f(y)) \leq C_f\, d(x,y)$$

(i.e. f is a d-contraction iff it reduces the $(d\text{-})$ distance between any pair of points in X)

* If f is a d-contraction then $f^{(m)}$ is a d-contraction $\forall m>0$. Proof: $\forall m>0$, $f^{(m)}$ is $(d/d$-$)$ Lipschitz continuous with $C_{f^{(m)}} = C_f^{m} < 1$.

(How do you think $f^{(m)}$ behaves when $m \to +\infty$ if f is a d-contraction?) ✓

* Let f be a self-map on X. $x^* \in X$ is a fixed point of f iff $x^* = f(x^*)$ (x^* remains invariant by f)

The issues of i. existence, ii. uniqueness, iii. location or "approximation" of fixed points is of great importance since it is obviously connected with the solution of generalized systems of equations. Fixed point theory is a corpus of results that deals with suchlike issues. We will henceforth deal with the Banach Fixed Point Theorem (BFPT) that uses the language of metric spaces.

* First id_X has fixed points. Trivially every $x \in X$ is a fixed point of id_X. Let f be a self map on X and x^* be a fixed point of f. Then x^* is also a fixed point of $f^{(m)}$ $\forall m \geq 0$ (the case of $m=0$ is trivial):

The claim is true for $m=1$ ($f^{(1)}=f$). Let x^* be a fixed point for $f^{(m)}$ when $m=k$ (i.e. $x^* = f^{(k)}(x^*)$) for some $k>0$. For $m=k+1$ we have $f^{(k+1)}(x^*) =$
$= f(f^{(k)}(x^*)) \overset{f^{(k)}(x^*)=x^*}{=} f(x^*) = x^*$.

* There is a partial converse to the above: if for some $m>0$, x^* is the unique fixed point of $f^{(m)}$ then it is also the unique fixed point of f. (Remark)

Proof. When $m=1$ the result is trivial $(f^{(1)}=f)$. Suppose that $m>1$, and let x^* be the unique fixed point of $f^{(m)}$.

We have that $f^{(m+1)}(x^*) = f(f^{(m)}(x^*)) = f(x^*)$

$$f^{(m+1)}(x^*) = f^{(m)}(f(x^*))$$

Hence $f^{(m)}(f(x^*)) = f(x^*)$. Hence $f(x^*) \in X$ is also a fixed point of $f^{(m)}$. But x^* is the unique fixed point of $f^{(m)}$. Hence $x^* = f(x^*)$ and thereby x^* is a fixed point of f.

Uniqueness of x^* as a fixed point of f: Say that $x^{**} \neq x^*$ is also a fixed point of f. But due to the previous remark $x^{**} = f^{(m)}(x^{**})$ which is impossible due to that $f^{(m)}$ has a unique fixed point.

[the previous remark is important: it could be easier in some cases to have info on the fixed points of $f^{(m)}$ for large enough m].

Banach Fixed Point Theorem (BFPT). Suppose that (X,d) is complete, and $f: X \to X$, is a d-contraction then f has a unique fixed point (say x^*), and $\forall x \in X$ for

$$x_n = \begin{cases} x, & n=0 \\ f(x_{n-1}), & n>0 \end{cases} = \begin{cases} x, & n=0 \\ f^{(n)}(x), & n>0 \end{cases} = f^{(n)}(x),$$

$$\lim_{n \to +\infty} x_n = x^* \text{ (w.r.t. } d).$$

Remark: BFPT is very powerfull: it asks a lot but it also gives uniqueness and approximation of the fixed point.

Proof. The proof will be based on the following lemmata:

Lemma (Uniqueness). If f is d-contraction and f has a fixed point then this is unique.
(this does not use completeness).

Proof. Suppose that x^*, x^{**} with $x^* \neq x^{**}$ are fixed points of f. Then

$$0 < d(x^*, x^{**}) = d(f(x^*), f(x^{**}))$$

$\leq C_f \, d(x^*, x^{**})$ where the last inequality follows due to the fact that f is a d-contraction.

Summarizing we have

$$0 < \underbrace{d(x^*, x^{**}) \leq c_f d(x^*, x^{**})}$$

$$\Downarrow$$

$$d(x^*, x^{**}) - c_f d(x^*, x^{**}) \leq 0 \iff$$

$(1-c_f) d(x^*, x^{**}) \leq 0$. This is impossible since

$1-c_f > 0$ and $d(x^*, x^{**}) > 0$. Hence f has a unique

fixed point. $\square$

Lemma (Fixed Points as Limits) If (x_n) (see above for the definition) has a limit w.r.t. d, say y, then y is a fixed point of f. (completeness is not used here)

Proof. We have that $y = \lim_{n\to+\infty} x_n = \lim_{n\to+\infty} f^{(n)}(x)$

w.r.t. d

$$= \lim_{n\to+\infty} f(f^{(n-1)}(x)) = \lim_{n\to+\infty} f(x_{n-1}) = f(\lim_{n\to+\infty} x_{n-1})$$

f is contraction

$\Downarrow$

Lipschitz cont.

$\Downarrow$

Cont.

$$\overset{x_n \to y}{\underset{x_{n-1}\to y}{=}} f(y) \Rightarrow y = f(y)$$

(why?)

hence $y = \lim_{n \to +\infty} x_n$ is a fixed point of f. $\square$

Lemma (Technical). $\forall n \geq 0 \quad d(x_{n+1}, x_n) \leq \underline{C_f}^n d(\underline{x_1, x_0})$

Proof. By induction:

a. It is true for $n=0$ since

$$d(x_1, x_0) = d(x_1, x_0) = C_f^0 d(x_1, x_0).$$

b. Suppose that it is true for $n=k$, i.e.

$$d(x_{k+1}, x_k) \leq C_f^k d(x_1, x_0) \quad (*).$$

c. For $n=k+1$ we have

$$d(x_{k+2}, x_{k+1}) = d(f(x_{k+1}), f(x_k)) \underset{f \text{ contraction}}{\leq}$$

$$C_f \underline{d(x_{k+1}, x_k)} \underset{(*)}{\leq} C_f \cdot C_f^k d(x_1, x_0)$$

$$= C_f^{k+1} d(x_1, x_0). \Rightarrow$$

$d(x_{k+2}, x_{k+1}) \leq C_f^{k+1} d(x_1, x_0)$ hence the property is true for $n=k+1$. $\square$

Lemma (Cauchyness). (x_n) is (d-) Cauchy.

Proof. We have to examine the limiting behavior of $d(x_m, x_n)$ as $\min(m,n) \to +\infty$.

Suppose first that $m > n$: due to triangle inequality

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_n)$$

$$\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + d(x_{m-2}, x_n)$$

$$\dots \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n)$$

$$\leq C_f^{m-1} d(x_1, x_0) + C_f^{m-2} d(x_1, x_0) + \dots + C_f^{n} d(x_1, x_0)$$

Technical

$$= C_f^n d(x_1, x_0) \left(C_f^{m-1-n} + C_f^{m-2-n} + \dots + 1 \right) =$$

$$= C_f^n d(x_1, x_0) \sum_{i=0}^{m-1-n} C_f^i \underset{C_f \geq 0}{\leq} C_f^n d(x_1, x_0) \sum_{i=0}^{\infty} C_f^i$$

$$\underset{0 \leq C_f < 1}{=} \frac{C_f^n}{1 - C_f} d(x_1, x_0)$$

$$\sum_{i=0}^{\infty} C_f^i = \frac{1}{1 - C_f}$$

geometric series

We have shown that if $m > n$, $d(x_m, x_n) \leq \frac{C_f^n}{1-C_f} d(x_1, x_0)$

It is obvious (by interchanging m with n) that

when $n \geq m$ $\quad d(x_m, x_n) \leq \dfrac{C_f^m}{1-C_f} d(x_1, x_0)$.

Hence we generally obtain that:

$$(**) \quad d(x_m, x_n) \leq \frac{C_f^{\min(m,n)}}{1 - C_f} d(x_1, x_0), \ \forall m, n \in \mathbb{N}$$

We have that due to $(**)$ and that $d(x_m, x_n) \geq 0$ $\forall m, n \in \mathbb{N}$

$$0 \leq \lim_{\min(m,n)\to+\infty} d(x_m,x_n) \leq \lim_{\min(m,n)\to+\infty} \frac{C_f^{\min(m,n)}}{1-C_f} d(x_1,x_0)$$

$$= \frac{d(x_1,x_0)}{1-C_f} \lim_{\min(m,n)\to+\infty} C_f^{\min(m,n)} \underset{0\leq C_f<1}{=} \frac{d(x_1,x_0)}{1-C_f} \cdot 0 = 0.$$

Hence (x_m) is d-Cauchy. □

Proof of BFPT continued: $\forall x \in X$, $x_m = f^{(m)}(x)$, (x_m) is a Cauchy sequence due Lemma Cauchyness. Since (X,d) is complete, (x_m) is (d-) convergent by being Cauchy. Due to Lemma Fixed points as limits, the (d-) limit of (x_m) is a fixed point of f. Due to Lemma Uniqueness f has a unique fixed point. Finally due to Lemma fixed points as limits, this unique fixed point is the limit of (x_m). □

Corollary. (X,d) is complete and for some $m>0$, $f^{(m)}$ is (d-) contraction. Then f has a unique fixed point.

Proof. By BFPT $f^{(m)}$ has a unique fixedpoint. Then due to Remark f has a unique fixed point. □

Example. Suppose that we want to study the equation (in $\mathbb{R}$)

$x = \sin(x)$. x is a solution iff x is a fixed point of the sin function. Hence we study the fixed point properties of this function. We know that $\mathbb{R}, d_u$ is complete. Is sin a d_u-contraction? Due to the results in the previous lecture sin is a d_u contraction iff $C_{\sin} := \sup_{x \in \mathbb{R}} \left| \frac{d\sin(x)}{dx} \right| < 1$

But $\sup_{x \in \mathbb{R}} |\cos(x)| = 1$ hence sin is d_u/d_u-Lipschitz continuous, but not a d_u-contraction. Consider $\sin^{(2)}(x) := \sin(\sin(x))$. Show that $C_{\sin^{(2)}} = \sup_{x \in \mathbb{R}} \left| \frac{d\sin(\sin(x))}{dx} \right| < 1$

Hence $\sin^{(2)}$ is contractive. Hence BFPT is applicable and thereb $\sin^{(2)}$ has a unique fixed point. Hence $\sin(x)$ has a unique fixed point, hence $x = \sin(x)$ has a unique solution. $\square$