Lecture 14/05/20

(Z, d_Z) a m.s.

Theorem (Convergence of (approximate) maximizers): Suppose that

i. $f_n, f \in B(Z, \mathbb{R})$ $\forall n \in \mathbb{N}$ and $d_{sup}(f_n, f) \to 0$

ii. $Z \ni x_f$ is the unique maximizer of f and $\forall \varepsilon > 0$ $\sup_{x \in O^c_{d_Z}(x_f, \varepsilon)} f(x) < f(x_f)$ (Uniqueness and distinguishibility)

iii. $\rho_n \geq 0$ with $\rho_n \to 0$.

Then if $x_{f_n, \rho_n} \in \rho_n\text{-}\underset{x \in Z}{\operatorname{argmax}}\, f_n(x)$, $d_Z(x_{f_n, \rho_n}, x_f) \to 0$.

Remark: Uniqueness and distinguishibility $\Rightarrow$ $\forall \varepsilon > 0$ and if $d_Z(x_n, x_f) > \varepsilon$ (i.e. $x_n \in O^c_{d_Z}(x_f, \varepsilon)$) finn $\Rightarrow$ $\exists \delta > 0$: $f(x_f) - f(x_n) > \delta$ finn.

$\hookrightarrow$ i.e. it is independent of the n's for which $d_Z(x_n, x_f) > \varepsilon$

The existence of δ is ensured by uniqueness. Its independence of the aforementioned n's is ensured by distinguishibility.

Proof. Suppose that the above is not true, i.e. $\exists$ $x_{f_n, \rho_n} \in \rho_n\text{-}\operatorname{argmax} f_n$ such that $d_Z(x_{f_n, \rho_n}, x_f) \not\to 0$, i.e. $\exists \varepsilon > 0$

$d_Z(X_{f_n,p_n}, X_f) > \varepsilon$ finn. $\overset{(ii)}{\Longrightarrow}$ $\exists\, \delta > 0$:

see Remark

$(*)$ $\underline{|f(X_{f_n,p_n}) - f(X_f)| > \delta}$. We have that:

$$|f(X_{f_n,p_n}) - f(X_f)| = |f(X_{f_n,p_n}) \pm f_n(X_{f_n,p_n}) - f(X_f)|$$

$$= |(f(X_{f_n,p_n}) - f_n(X_{f_n,p_n})) + (f_n(X_{f_n,p_n}) - f(X_f))|$$

$$\overset{\text{tr.in}}{\leq} |f(X_{f_n,p_n}) - f_n(X_{f_n,p_n})| + |f_n(X_{f_n,p_n}) - f(X_f)|$$

$$\leq \sup_{x \in Z} |f(x) - f_n(x)| + \underbrace{\sup_{x \in Z} |f_n(x) - f(X_f)| + p_n}$$

from the proof of the previous theorem

$$= 2\, d_{\sup}(f_n, f) + p_n. \quad (**)$$

Hence by $(*)$, $(**)$ we have that

$$2\, d_{\sup}(f_n, f) + p_n > \delta \text{ finn.}$$

Since $d_{\sup}(f_n, f) \to 0$, and $p_n \to 0$ $\Rightarrow$ $2\, d_{\sup}(f_n, f) + p_n \to 0$.

Hence it is impossible for $\underline{2\, d_{\sup}(f_n, f) + p_n > \delta \text{ finn}}$ to hold. $\square$

Remark: The above is quite general. It allows for the existence of optimization errors (ξ_n) due to computational procedures of optimization, among others.

Exercise: State and prove the dual form of the above concerning minimization.

Further Non topological notions in Metric Spaces

1. Completeness

A. Cauchy Sequences (Βασικές Ακολουθίες)

Definition. (X,d) a m.s. and (x_n) is such that $x_n \in X \ \forall n \in \mathbb{N}$. Then (x_n) is Cauchy iff $\forall \varepsilon > 0$, $\exists n^*(\varepsilon)$: $\forall n, m \geq n^*(\varepsilon)$ $d(x_n, x_m) < \varepsilon$.

Remark: The above describes a property of "asymptotic concentration" for (x_n). $\forall \varepsilon > 0 \ \exists n^*(\varepsilon)$:

distance $< \varepsilon$

$$(x_0, x_1, x_2, \ldots, \underset{=}{x_{n^*(\varepsilon)}}, x_{n^*(\varepsilon)+1}, \ldots, x_n, x_m, \ldots)$$

distance $< \varepsilon$ distance $< \varepsilon$

etc.

Example. $X = \mathbb{R}$, $d = d_u$ $\quad x_n = \frac{1}{n+1}$. Is (x_n) Cauchy?

We examine $(*)$ $|x_n - x_m| < \varepsilon \iff \left|\frac{1}{n+1} - \frac{1}{m+1}\right| < \varepsilon$. Let $m^* = \max(n,m)$

$n^* = \min(n,m) \Rightarrow m^* - n^* = k \in \mathbb{N} \iff m^* = n^* + k$.

$$\left|\frac{1}{n+1} - \frac{1}{m+1}\right| = \frac{1}{n^*+1} - \frac{1}{m^*+1} = \frac{m^*+1 - (n^*+1)}{(n^*+1)(m^*+1)} = \frac{m^* - n^*}{(n^*+1)(n^*+1+k)} =$$

$$= \frac{k}{(n^*+1)^2 + k(n^*+1)}.$$

$$(*) \iff \frac{k}{(n^*+1)^2 + k(n^*+1)} < \varepsilon \iff \frac{k}{\varepsilon} < (n^*+1)^2 + k(n^*+1) \iff$$

$$k(n^*+1) + (n^*+1)^2 > \frac{k}{\varepsilon} \quad (**)$$ (when $k=0$ $n=m$ hence $(*)$ is in any case valid)

Hence suppose $k > 0$, hence $(**) \iff n^*+1 + \frac{(n^*+1)^2}{k} > \frac{1}{\varepsilon}$

$\iff n^* + \cancel{\frac{(n^*+1)^2}{k}} > \frac{1}{\varepsilon} + 1$. $(***)$. The $(***)$ is valid whenever $n^* > \frac{1}{\varepsilon} - 1$ and the latter holds

$\forall n^* \in \mathbb{N}$: $n^* \geq$ smallest natural greatest than or equal to $\frac{1}{\varepsilon} - 1$

Hence if we define $n^*(\varepsilon) :=$ smallest natural $\geq \frac{1}{\varepsilon} - 1$ we have that $\forall n, m \geq n^*(\varepsilon)$ $\quad |x_n - x_m| < \varepsilon$. Since ε is arbitrary this implies that $\left(\frac{1}{n+1}\right)$ is Cauchy. $\square$

Counter-Example. $X = \mathbb{R}$, $d = d_u$, $x_n = n$. Is (n) Cauchy?

Suppose that it is. Let $\varepsilon = 1/3$. Then $\exists n^*(1/3) : \forall n, m \geq n^*(1/3)$

$|x_n - x_m| < 1/3$. Set $n := n^*(1/3)$, $m := n^*(1/3) + 1$. So we must

have that $|x_n - x_m| = |n^*(1/3) - (n^*(1/3) + 1)| = 1$ and

$1 < 1/3$, impossible. Hence (n) is not Cauchy. $\square$

Lemma. If (x_n) converges then it is Cauchy. (w.r.t. d)

Proof. Let $\varepsilon > 0$. Let x be the limit of (x_n). $\exists\, n(\varepsilon/2)$:

$\forall n \geq n(\varepsilon/2)$ $d(x, x_n) < \varepsilon/2$. $\forall n, m \geq n(\varepsilon/2)$, $d(x_n, x_m) \leq \underbrace{d(x, x_n)}_{<\varepsilon/2} + \underbrace{d(x, x_m)}_{<\varepsilon/2}$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $\forall n, m \geq n(\varepsilon/2)$, $d(x_n, x_m) < \varepsilon$. Set $n^*(\varepsilon) := n(\varepsilon/2)$.

The result follows since ε is arbitrary. $\square$

Remark. We have shown that the set of convergent sequences $\subseteq$ the set of Cauchy sequences in (X, d).

The reverse implication does not hold as the following example shows:

Example. $X = (0, 1]$, $d = d_u$ restricted to $(0, 1]$. $x_n = 1/_{n+1} \in (0, 1]$

$\forall n$. ((x_n) also belongs to the particular METRIC SUBSPACE of $(\mathbb{R}, d_u)$).

It is easy to see that $(1/_{n+1})$ is Cauchy as a sequence

inside $(0, 1]$ — use the exact same arguments with the first example.

But $x_n \to 0 \notin X$, hence (x_n) cannot be considered as convergent in the particular X. Hence it is possible that a metric space (X,d) "does not contain the limits" of some of its Cauchy sequences. Hence it is possible that the set of convergent sequences $\subsetneq$ the set of Cauchy sequences in (X,d).

B. Complete Metric Spaces.

Definition. (X,d) is complete iff every Cauchy sequence in X (w.r.t. d) is also convergent in X (w.r.t. d)

[i.e. iff the set of convergent sequences = the set of Cauchy sequences in (X,d)].

Example. It can be shown that $(\mathbb{R}, d_u)$ is complete.

Counter Example. $([0,1], \underset{\text{restriction}}{d_u})$ is not complete as the Previous example shows.

Example. X is general $d = d_d$. (x_n) is Cauchy (w.r.t. d_d) iff it is eventually constant. (Suppose that it is eventually constant i.e. it is of the form $(x_0, x_1, \dots, x_k, c, c, c, \dots, c, \dots)$ for some $k \in \mathbb{N}$ and $c \in X$. $\forall \varepsilon > 0$ $d(x_n, x_m) = 0 < \varepsilon$ $\forall n, m \geq k+1$, hence it is Cauchy) (Suppose that (x_n) is (d_d)-Cauchy then for $\varepsilon = 1/2$, $\exists n^*(1/2): \forall n,m \geq n^*(1/2), d_d(x_n, x_m) < 1/2$

$\Leftrightarrow x_n = x_m \;\forall n,m \geq n^*(1/2)$ hence (x_n) is eventually constant). We have already proven that (x_n) is $(d_d\text{-})$ convergent iff it is eventually constant. Hence

Set of Cauchy Sequences = Set of eventually constant sequences

= Set of convergent sequences in (X, d_d).

Hence every discrete space is complete!

(Remember Cauchyness and Completeness also depend on the metric!)

When is the completeness property inherited by metric subspaces?

Let (X,d) is a complete metric space. $A \subseteq X$, we consider the metric subspace (A,d). The subspace is complete iff A is a closed subset of X.

$\not\phi$

restriction

Proof. (if (A,d) is complete then A is closed) Let (x_n) be such that $x_n \in A \;\forall n \in \mathbb{N}$, and $x_n \to x \in X$. Since (x_n) is convergent in X it is Cauchy in X. Hence it is Cauchy in A. Since (A,d) is complete (x_n) converges in A. Hence by the uniqueness of limits $x \in A$. Since (x_n) is arbitrary A is closed.

(if A is closed then (A,d) is complete). Suppose that (x_n) is Cauchy in A. Hence (x_n) is Cauchy in X. Since (X,d) is complete then (x_n) converges in X. Since A is closed (x_n) converges in A. Since (x_n) is arbitrary every Cauchy sequence in A converges in A. Hence (A,d) is complete. $\square$

Example. Let $X = B(Z,Y)$, (Y,d_Y), $d^Y_{\sup}(f,g) = \sup_{x\in Z} d_Y(f(x),g(x))$. It is possible to show that if Y, d_Y is complete then $(B(Z,Y), d^Y_{\sup})$ is complete. We know that $(\mathbb{R}, d_u)$ is complete then by the previous shows that $(B(Z,\mathbb{R}), d_{\sup})$ is complete. (see tutorial for the proof).

C. Cauchyness – Completeness and Metrics Comparison

X general, d_1, d_2 metrics and $\exists c>0 : d_1 \leq c d_2$.

- If (x_n) is Cauchy w.r.t. d_2 then it is Cauchy w.r.t. d_1 (Exercise)
- Hence set of Cauchy sequences in $(X,d_2) \subseteq$ Set of Cauchy sequences in (X,d_1) $\square$