Lecture 12/05/20

- We have shown that if $\exists c>0: d_1 \leq c d_2$, and $x_n \to x$ w.r.t. d_2 $\Rightarrow$ $x_n \to x$ w.r.t. d_1

(hence the set of convergent in X sequences w.r.t. d_2 $\subseteq$ to the set of convergent in X sequences w.r.t. d_1) //

- Thereby, (why), if $\exists c_1, c_2 > 0$: $c_1 d_2 \leq d_1 \leq c_2 d_2$ then
$x_n \to x$ w.r.t. d_1 $\Leftrightarrow$ $x_n \to x$ w.r.t. d_2

(hence the aforementiones sets coincide)

E.g. $X = \mathbb{R}^p$, $d = d_I, d_{max}, d_{11}$ are not "distinguishable" w.r.t. sequential convergence.

Closeness

Definition: $A \subseteq X$ is closed (w.r.t. d) iff for any sequence of elements of A $\left((x_n), x_n \in A \ \forall n \in \mathbb{N} \right)$ that has a d-limit, its limit lies in A (i.e. every convergent sequence of elements of A converges inside A).

Dually: $A \subseteq X$ is open (w.r.t. d) iff A^c is closed.

- It is easy to show that: i. $X, \emptyset$ are simultaneously open and closed.

ii. If τ_c denotes the collection of the closed subsets of X then $X, \emptyset \in \tau_c$, arbitrary intersections of elements of τ_c lie in τ_c (i.e. arbitrary intersections of closed subsets of X are closed.

ii. (dual) If τ_o denotes the collection of the open subsets of X, then $X, \phi \in \tau_o$, arbitrary unions of elements of τ_o lie in τ_o (i.e. arbitrary intersections of open subsets of X are open).

τ_o is term topology on X generated by d.

E.g. It is possible to show that for any $x \in X$ and $\varepsilon > 0$ $O_d(x, \varepsilon)$ is open (i.e. $\in \tau_o$), and $\{x\}$, $O_d[x, \varepsilon]$ are closed (i.e. $\in \tau_c$).

Continuity of functions between Metric Spaces.

Suppose $f : X \to Y$, (X, d_X), (Y, d_Y) are Metric spaces.

- Suppose that (x_n) is a X-sequence (i.e. $x_n \in X$, $\forall n \in \mathbb{N}$).

By evaluation via f we obtain the Y-sequence $(f(x_n))$.

Continuity of f at a point involves the comparison of such sequences:

Definition. Let $x \in X$, f is (d_Y/d_X-) continuous at x iff $\forall (x_n)$, with $x_n \in X$ $\forall n \in \mathbb{N}$, that converges (w.r.t. d_X) to x, $(f(x_n))$ converges (w.r.t. d_Y) to $f(x)$. (f must not destroy the convergence of sequences in X that converge to x).

f is (d_Y/d_X-) continuous iff it is (d_Y/d_X-) continuous at x, $\forall x \in X$.

E.g. X, $d_X = d_d$ (discrete metric). We know that $x_n \to x$ w.r.t. d_d iff (x_n) is eventually constant at x (i.e. it is of the form $(x_0, x_1, \dots, x_k, x, x, x, \dots, x, \dots)$ for some k).

We then have that $(f(x_n))$ is eventually constant at $f(x)$ (since it is of the form $(f(x_0), f(x_1), \dots, f(x_k), f(x), \dots, f(x), \dots, f(x), \dots)$)

Thereby $f(x_n) \to f(x)$ w.r.t. d_Y (for any d_Y)

Hence if (X, d_X) is discrete EVERY function $X \to Y$ (with whatever d_Y) is continuous

Remark. We can characterize continuity at x using appropriately open (or closed) balls, or more generally using open (or closed) sets (see tutorial). For example it is possible to show that f is continuous at x iff

$$\forall \delta > 0, \exists \varepsilon > 0 : \underbrace{f^{-1}(O_{d_Y}(f(x), \delta))}_{\{x \in X : f(x) \in O_{d_Y}(f(x), \delta)\}} \subseteq O_{d_X}(x, \varepsilon).$$

Application: Approximation of Optimization Problems

- $X = B(Z, \mathbb{R})$ (or some suitable subset), $d_X = d_{sup}$

  $Z \neq \emptyset$

- $Y = \mathbb{R}$, $d_Y = d_u$

- $\sup : B(Z, \mathbb{R}) \to \mathbb{R}$ (for any $g \in B(Z, \mathbb{R})$ $\sup(g)$ exists as a real number hence sup is well defined as a function from $B(Z, \mathbb{R}) \to \mathbb{R}$). The evaluation of sup at $g \in B(Z, \mathbb{R})$ is equivalent to the Maximization of g.

- Is sup (possibly restricted to some subset of $B(Z, \mathbb{R})$) d_u / d_{sup} - continuous? (Using the definition above this is equivalent to asking whether $\forall f_n, f \in B(Z, \mathbb{R})$ such that $d_{sup}(f_n, f) \to 0$, $\sup f_n \to \sup f$ w.r.t. d_u, i.e. $|\sup f_n - \sup f| \to 0$.)

  If this is true then we can "approximate" the issue of Maximization of f by the Maximization of f_n.

- Dually we can work with inf (Exercise!!!)

- Let $g \in B(Z, \mathbb{R})$. It is possible that even though $\sup(g)$ is well-defined, that $\max g$ (i.e. $\max_{x \in Z} g(x)$)

$\sup(g) = \sup_{x \in Z}(g(x))$ does not exist due to that $\arg\max_{x \in Z} g(x)$ is empty.

However it is possible to prove that: if $\rho > 0$, $\exists x_{g,\rho} \in Z$:

$$\underline{g(x_{g,\rho}) \geq \sup_{x \in Z} g(x) - \rho} = \sup_{x \in Z}(g(x) - \rho)$$

ρ can be perceived as an optimization error.

$x_{g,\rho}$ is an approximate maximizer

We denote the set of ρ-approximate maximizers of g with $\rho\text{-}\arg\max_{x \in Z} g(x)$.

* Obviously if we let $\rho = 0$ then $\rho\text{-}\arg\max_{x \in Z} g(x) = \arg\max_{x \in Z} g(x)$ when it exists. Moreover if we let Z be a Metric space that is also totally bounded and complete (next lecture) then it is possible to prove that $\exists g \in B(Z, \mathbb{R})$ such that $\rho\text{-}\arg\max g$ exist even for $\rho = 0$.

- We keep the fact that when $\rho > 0$, $\rho\text{-}\arg\max_{x \in Z} g(x) \neq \emptyset$ $\forall g \in B(Z, \mathbb{R})$.

Theorem. Given the above the $\sup : B(Z,\mathbb{R}) \to \mathbb{R}$ function is $\left(d_u / d_{\sup} -\right)$ continuous.

Proof. Suppose that continuity is not the case. This is equivalent to the existence of $f \in B(Z,\mathbb{R})$ and $f_n \in B(Z,\mathbb{R})$ such that $d_{\sup}(f_n, f) = \sup_{x\in Z} |f_n(x) - f(x)| \to 0$ (as $n \to +\infty$) yet $d_u(\sup f_n, \sup f) = |\sup_{x\in Z} f_n(x) - \sup_{x\in Z} f(x)| \not\to 0$.

The latter means that $\exists\, \delta > 0$:

$$|\sup_{x\in Z} f_n(x) - \sup_{x\in Z} f(x)| > \delta \text{ for an infinite number of } n \quad (*)$$

[finin?]

Let $p_n > 0$ such that $p_n \to 0$ as $n \to +\infty$ (e.g. $p_n = \frac{1}{n+1}$) ✓

Since $p_n > 0$ $\forall n$, we have that p_n-argmax f_n and p_n-argmax f are not empty. $x_{f_n, p_n} \in p_n$-argmax(f_n), $x_{f, p_n} \in p_n$-argmax(f).

Consider $|\sup f_n(x) - \sup f(x)| = |(\underbrace{\sup f_n(x) - p_n}_{A_n}) - (\underbrace{\sup f - p_n}_{B_n})|$

We examine $|A_n - B_n|$:

α. $A_n \geq B_n$ $\quad |A_n - B_n| = A_n - B_n =$

$$= (\sup f_n(x) - p_n) - \underbrace{(\sup f - p_n)}_{\searrow 0} \leq$$

$$f_n(\underline{\underline{x_{f_n,p_n}}}) - (\sup f - p_n) =$$

$$f_n(x_{f_n,p_n}) \pm f(x_{f_n,p_n}) - (\sup f - p_n) =$$

$$= \left(f_n(x_{f_n,p_n}) - f(x_{f_n,p_n})\right) + \cancel{\left(f(x_{f_n,p_n}) - \sup f\right)} + p_n$$

$$\leq \left(f_n(x_{f_n,p_n}) - f(x_{f_n,p_n})\right) + p_n$$

$$= \left| f_n(x_{f_n,p_n}) - f(x_{f_n,p_n}) + p_n \right|$$

tr.

$$\leq \left| f_n(x_{f_n,p_n}) - f(x_{f_n,p_n}) \right| + p_n$$

$$\leq \sup_{x \in Z} \left| f_n(x) - f(x) \right| + p_n = d_{sup}(f_n, f) + p_n$$

b. $B_n > A_n \Rightarrow |A_n - B_n| = B_n - A_n$

then using a similar reasoning (exercise – swap in the previous f for f_n, f_n for f, and x_{f,p_n} for x_{f_n,p_n})

we have that $B_n - A_n \leq d_{sup}(f_n, f) + p_n$

Hence by a, b we have that:

$$|\sup f_n - \sup f| \leq \underline{d_{sup}(f_n, f) + p_n} \quad \forall n \in \mathbb{N}$$

hence $(*) \Rightarrow \cancel{d_{sup}(f_n, f)} \cancel{+ p_n} > \delta$ fine if this is

true this implies that $d_{sup}(f_n, f) + p_n \not\to 0$

which is impossible since $d_{sup}(f_n, f) \to 0$ and $p_n \to 0$. $\square$

Remark: A dual result holds for the inf function (Exe!!!)

Remark. The previous imply that uniform convergence implies the approximability of optimization problems.
(it thus pays to study uniform convergence!!!)

What about optimizers?

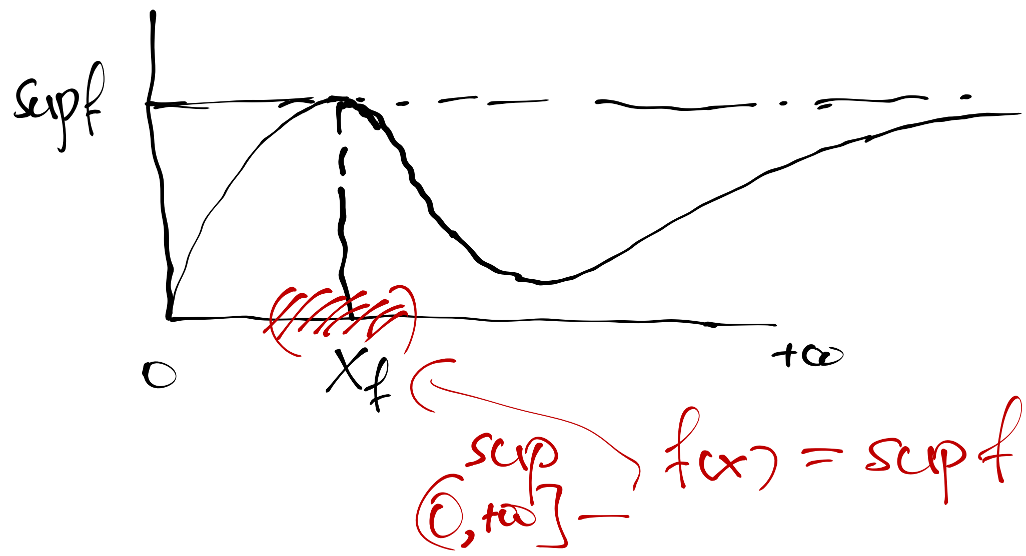
In order to study the issue of approximation of the (approximate) optimizers (those belong to Z) we need to be able to examine convergence in Z. Hence we now assume that Z, d_Z is a metric space. We assume the following:

i. $f_n, f \in B(Z, \mathbb{R})$, $\forall n \in \mathbb{N}$ such that $d_{sup}(f_n, f) \to 0$.

ii. There exists a unique $x_f \in Z$ such that $\{x_f\} = \arg\max_{x \in Z} f(x)$ and furthermore, x_f is "distinguishable" which means that, $\forall \varepsilon > 0$

$$\sup_{x \in O^c_{d_Z}(x_f, \varepsilon)} f(x) < f(x_f) = \sup_{x \in Z} f(x)$$

the distinguishability assumption precludes behaviours like the following:



to be continued...