Tuesday 31/03

1. We have been occupied with the notion of boundness on $B(Y,\mathbb{R})$, d_{sup}:
If $A \subseteq B(Y,\mathbb{R})$ then A is uniformly bounded iff $\sup_{f\in A} \sup_{x\in Y} |f(x)| < +\infty$.
We have proven that A uniformly bounded $\Leftrightarrow$ A is a bounded subset of $B(Y,\mathbb{R})$ w.r.t. the uniform metric (d_{sup})

We have already seen a counterexample:

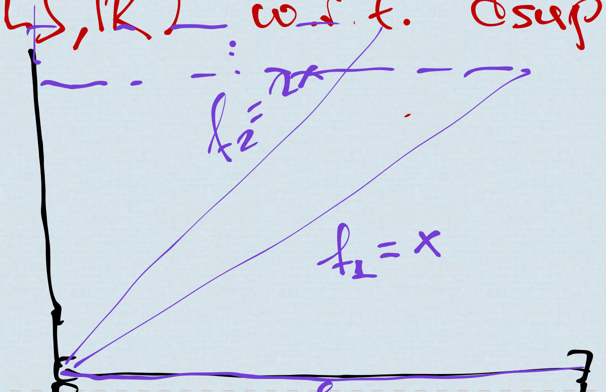
$Y = [0,1]$ $\qquad A = \{ f_n : [0,1] \to \mathbb{R},\ f_n(x) = nx,\ n\in\mathbb{N}\}$

We have shown that $\forall n\in\mathbb{N}$ $f_n \in B([0,1],\mathbb{R})$,
since $\sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} |nx| = \sup_{x\in[0,1]} nx = n \sup_{x\in[0,1]} x = n < +\infty$
$\forall n\in\mathbb{N}$. Hence $A \subseteq B([0,1],\mathbb{R})$. Is it bounded w.r.t. d_{sup}?

We used Uniform boundness:

$$\sup_{f_n\in A} \sup_{x\in[0,1]} |f_n(x)| = \sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} |f_n(x)| = \dots =$$

$$= \sup_{n\in\mathbb{N}} n = +\infty$$, hence A is not uniformly bounded and thereby A is <u>not</u> a bounded subset of $B([0,1],\mathbb{R})$ w.r.t. d_{sup}.



Example: $Y=[0,1]$, $A=\{f_n:[0,1]\to\mathbb{R},\ f_n(x)=\frac{x}{n+1}, n\in\mathbb{N}\}$

Is $A\subseteq B([0,1],\mathbb{R})$? We have that

$$\sup_{x\in[0,1]}|f_n(x)| = \sup_{x\in[0,1]}\left|\frac{x}{n+1}\right| = \sup_{x\in[0,1]}\frac{x}{n+1}$$

$$\frac{1}{n+1}\sup_{x\in[0,1]} x = \frac{1}{n+1} < +\infty \quad \forall n\in\mathbb{N} \Rightarrow f_n\in B([0,1],\mathbb{R})$$

$\forall n\in\mathbb{N} \Leftrightarrow A\subseteq B([0,1],\mathbb{R})$. Hence we are entitled to ask if A is bounded w.r.t. d_{sup}.

We will use uniform boundness:

$$\sup_{f_n\in A}\sup_{x\in[0,1]}|f_n(x)| = \sup_{n\in\mathbb{N}}\sup_{x\in[0,1]}\left|\frac{x}{n+1}\right|$$

$$= \sup_{n\in\mathbb{N}}\sup_{x\in[0,1]}\frac{x}{n+1} = \sup_{n\in\mathbb{N}}\left[\frac{1}{n+1}\sup_{x\in[0,1]} x\right] = \sup_{n\in\mathbb{N}}\left[\frac{1}{n+1}\cdot 1\right]$$

$$= \sup_{n\in\mathbb{N}}\frac{1}{n+1} = 1 < +\infty$$ hence A is uniformly bounded and thereby A is a bounded subset of $B([0,1],\mathbb{R})$ w.r.t. d_{sup}.

Example: $Y=[0,1]$, $L>0$

$$A=\{f:[0,1]\to\mathbb{R},\ f(0)=0,\ \forall x,y\in[0,1]\ |f(x)-f(y)|\le L|x-y|\}$$

Is $A \neq \emptyset$? Consider $f(x) = \frac{L}{e}(e^x - 1)$, $x \in [0,1]$

We have that $f(0) = \frac{L}{e}(e^0 - 1) = \frac{L}{e}(1 - 1) = 0$

For the second property we have that:

Let $x, y \in [0,1]$, we have that due to the Mean Value theorem that

$$f(x) = f(y) + \frac{\partial f}{\partial x}(x^*)(x-y), \text{ where}$$

x^* is "between" x and y, we also have that

$\frac{\partial f}{\partial x}(x^*) = \frac{L}{e} e^{x^*}$ and thereby due to the previous

$$f(x) - f(y) = \frac{L}{e} e^{x^*}(x-y) \Rightarrow$$

$$|f(x) - f(y)| = \left|\frac{L}{e} e^{x^*}\right| |x-y| \Leftrightarrow$$

$$|f(x) - f(y)| = \frac{L}{e} e^{x^*} |x-y| \Rightarrow$$

↳ maximise w.r.t. $x^* \in [0,1]$

$$|f(x) - f(y)| \leq \frac{L}{e} \sup_{x^* \in [0,1]} e^{x^*} |x-y| \Leftrightarrow$$

$$|f(x) - f(y)| \leq \frac{L}{e} e |x-y| = L|x-y|$$

Hence this particular $f \in A$ (Exercise: Has A other members?)

Is $A \subseteq B([0,1], \mathbb{R})$? Let $f \in A$

$$\sup_{x\in[0,L]} |f(x)| = \sup_{x\in[0,L]} |f(x)-0|$$

$$= \sup_{x\in[0,L]} |f(x)-f(0)| \leq \sup_{x\in[0,L]} \left[L\,|x-0| \right]$$

$$= \sup_{x\in[0,L]} L\,|x| = L \sup_{x\in[0,L]} x = L\cdot L = L < +\infty$$

hence $f\in B([0,L],\mathbb{R}) \xrightarrow[f \text{ arbitrary}]{} A \subseteq B([0,L],\mathbb{R})$

Hence we are entitled to ask whether it is a bounded subset of $B([0,L],\mathbb{R})$ w.r.t. $d_{\sup}$.

We use uniform boundness:

$$\sup_{f\in A} \sup_{x\in[0,L]} |f(x)| = \sup_{f\in A} \sup_{x\in[0,L]} |f(x)-0|$$

$$= \sup_{f\in A} \sup_{x\in[0,L]} |f(x)-f(0)| \leq \sup_{f\in A} \sup_{x\in[0,L]} \left(L\,|x-0| \right)$$

L is independent of f (it describes a property that holds uniformly w.r.t. the elements of A)

$$= L \sup_{f\in A} \underbrace{\sup_{x\in[0,L]} |x|}_{=L} = L \sup_{f\in A} L = L < +\infty$$

hence A is uniformly bounded and hence it is a bounded subset of $B([0,L],\mathbb{R})$ w.r.t. $d_{\sup}$.

Exercise. In the previous frameworks let now

$c \in \mathbb{R}, L > 0$ $\quad A = \{ f : [0,L] \to \mathbb{R},\ f(0) = c,\ \forall x,y \in [0,L]$

$|f(x) - f(y)| \leq L\,|x-y| \}$

Is A a bounded subset of $B([0,L],\mathbb{R})$ w.r.t $d_{\sup}$?

Example. $Y = \mathbb{N}$,

Lets think about the representation of a function $f : \mathbb{N} \to \mathbb{R}$; this should be a function that when evaluated at any natural number it gives a real number;

$f(0), f(1), f(2), \dots, f(n), \dots \in \mathbb{R}$

$$(f(0), f(1), f(2), \dots, f(n), \dots)$$ hence

f can be equivalently represented as a real vector that has a first element, does not have a last element, and has as many elements as the cardinality of $\mathbb{N}$.

Hence any $f : \mathbb{N} \to \mathbb{R}$ is representable by a vector with the previous properties.

Those vectors are called real sequences.

A real sequence is bounded iff it is a bounded function $f: \mathbb{N} \to \mathbb{R}$

Example: $f: \mathbb{N} \to \mathbb{R}$, $f(n) = n$, $n \in \mathbb{N}$

$(0, 1, 2, \dots, n, \dots)$

Is this real sequence bounded? $\sup_{n\in\mathbb{N}} |f(n)| = \sup_{n\in\mathbb{N}} |n|$

$= \sup_{n\in\mathbb{N}} n = +\infty$, hence this real sequence is unbounded.

Example. $f: \mathbb{N} \to \mathbb{R}$, $f(n) = \frac{1}{n+1}$

$(1, 1/2, 1/3, \dots, 1/n+1, \dots)$ Is f bounded?

$\sup_{n\in\mathbb{N}} |f(n)| = \sup_{n\in\mathbb{N}} |\frac{1}{n+1}| = \sup_{n\in\mathbb{N}} \frac{1}{n+1} = 1 < +\infty$. hence it is bounded.

Hence $B(\mathbb{N}, \mathbb{R})$ = set of bounded real sequences equiped with d_{sup} is well defined.

Consider $A = \left\{ f_m : \mathbb{N} \to \mathbb{R},\ f_m(n) = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases},\ m \in \mathbb{N} \right\}$

e.g. $f_0(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$

$(1, 0, 0, \dots, 0, \dots)$

$$f_1(n) = \begin{cases} 0 & n \neq 1 \\ 1 & n = 1 \end{cases}$$

$$(0, 1, 0, \dots, 0, \dots)$$

$$f_2(n) = \dots$$

$$(0, 0, 1, 0, \dots, 0, \dots)$$

$$\vdots$$

Is $A \subseteq B([0,1], \mathbb{R})$?

Consider f_m for arbitrary $m \in \mathbb{N}$,

$$\sup_{n \in \mathbb{N}} |f_m(n)| = 1 < +\infty \implies f_m \in B(\mathbb{N}, \mathbb{R})$$

$$\xRightarrow{m \text{ arbitrary}} A \subseteq B(\mathbb{N}, \mathbb{R}).$$

Hence we are entitled to ask whether A is a bounded subset of $B(\mathbb{N}, \mathbb{R})$ w.r.t. d_{sup}.

We are again using uniform boundness:

$$\sup_{f_m \in A} \sup_{n \in \mathbb{N}} |f_m(n)| = \sup_{m \in \mathbb{N}} \sup_{n \in \mathbb{N}} |f_m(n)|$$

$$= \sup_{m \in \mathbb{N}} \sup_{n \in \mathbb{N}} \max(0, 1) = 1 < +\infty$$

Hence A is uniformly bounded and thereby A is a bounded subset of $B(\mathbb{N}, \mathbb{R})$ w.r.t. d_{sup}. $\square$

Return to generality:

let again X is a reference set that can be equiped with d_1, d_2. Suppose that for some $c>0$ we have that $d_1 \leq c\, d_2$

(functional relation: $\forall x, y \in X \quad d_1(x,y) \leq c\, d_2(x,y)$)

($\Rightarrow$ $x \in X$, $\varepsilon > 0$ $\quad O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x, c\varepsilon)$ (*)

$O_{d_2}[x,\varepsilon] \subseteq O_{d_1}[x, c\varepsilon]$)

Suppose $A \subseteq X$ is bounded w.r.t. d_2. $\Leftrightarrow$ $\exists x \in X, \varepsilon > 0$

$$A \subseteq O_{d_2}(x,\varepsilon) \overset{(*)}{\subseteq} O_{d_1}(x, c\varepsilon)$$

Hence A is also bounded w.r.t. d_1.

Thereby if A is bounded w.r.t. the dominant metric it is also bounded w.r.t. the dominated metric.