Lecture 5

- Sub-Examples of Uniform Boundedness
- Boundedness and Metric Subspaces
- Boundedness lost in the limit
- Function Spaces (again)
- Another look at the definition
- Total boundedness

* Uniform boundedness: an analytically convenient way to characterize (d_{sup}-) boundedness.

▸ Remember that: we work in $(B(Y,\mathbb{R}), d_{sup})$.

- $A \subseteq B(Y,\mathbb{R})$ is uniformly bounded iff $\sup_{f\in A, x\in Y} |f(x)| < +\infty$
- A d_{sup}-bounded iff A uniformly bounded. ∎

Subexamples:

1. $Y=\mathbb{N}$, $B(\mathbb{N},\mathbb{R})$ space of bounded real sequences equipped with the uniform Metric. e_i is represented as a function $\mathbb{N}\to\mathbb{R}$: $e_i(n)=\begin{cases}1, & i=n\\ 0, & i\neq n\end{cases}$

Let $A := \{ e_i = (0,0,\dots,1,0,\dots 0,\dots),\ i\in\mathbb{N}\}$

↳ i^{th} position

Then $\sup_{f\in A}\sup_{x\in Y} |f(x)| = \sup_{i\in\mathbb{N}}\sup_{n\in\mathbb{N}} |e_i n|$

$= \sup_{i\in\mathbb{N}}\sup_{n\in\mathbb{N}} \left|\begin{cases}0 & i\neq n\\ 1 & i=n\end{cases}\right| = \sup_{i\in\mathbb{N}}\sup_{n\in\mathbb{N}} 1 = 1 < +\infty$

Hence A uniformly bounded $\Leftrightarrow$ d_{sup} bounded subset of $B(\mathbb{N},\mathbb{R})$. ∎

2. $X=[0,1]$, $B([0,1],\mathbb{R})$, d_{sup}

e.g. $x\in[0,1]$, $g(x)=Lx\in A_L$

$L>0$, $A_L=\{f:[0,1]\to\mathbb{R},\ f(0)=0,\ \forall x,y\in[0,1],\ |f(x)-f(y)|\le L|x-y|\}$

We have proven that $\emptyset\ne A_L\subseteq B([0,1],\mathbb{R})$

Is it uniformly bounded?

$$\sup_{f\in A_L}\sup_{x\in[0,1]}|f(x)| = \sup_{f\in A_L}\sup_{x\in[0,1]}|f(x)-\overset{=0}{f(0)}| \le$$

$$\sup_{f\in A_L}\sup_{x\in[0,1]}L|x| = L\sup_{x\in[0,1]}x = 1\cdot L \overset{=L}{<} +\infty \Rightarrow A_L \text{ unif. bounded}$$

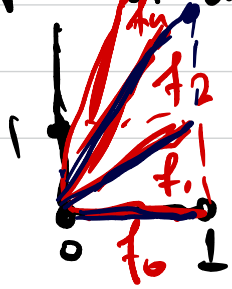
$\Leftrightarrow$ A_L d_{sup}-bounded.

3. (Counter-) Sub-example: The previous framework but

$A=A_{\mathbb{N}}=\{f_n:[0,1]\to\mathbb{R},\ f_n(x)=nx,\ n\in\mathbb{N}\}\subseteq B([0,1],\mathbb{R})$

$$\sup_{f\in A_{\mathbb{N}}}\sup_{x\in[0,1]}|f(x)|=\sup_{n\in\mathbb{N}}\sup_{x\in[0,1]}|f_n(x)|=\sup_{n\in\mathbb{N}}\sup_{x\in[0,1]}|nx|=\sup_{n\in\mathbb{N}} n\overset{=1}{\sup_{x\in[0,1]}x}$$

$=\sup_{n\in\mathbb{N}} n=+\infty \Rightarrow A_{\mathbb{N}}$ is not Uniformly bounded $\Rightarrow A_{\mathbb{N}}$ is not d_{sup}-bounded



$\forall n\in\mathbb{N}$ $f_n([0,1]) = [0,+\infty)$ Not a bounded subset of $\mathbb{R}$

$= [0,n]$

$n=0$

[redacted]

We have already seen this

Further details on boundness

1. X, d_1, d_2, $A \subseteq X$, $d_1 \le c d_2$, $c > 0$.

If A is bounded w.r.t. d_2 then A is also bounded w.r.t. d_1.

Corollary. Suppose that $\exists\, c_1, c_2 > 0 : c_1 d_2 \le d_1 \le c_2 d_2$ ✓ (*)

$\Updownarrow$

(– $d_1 \le c^* d_2$ for
– $d_2 \le c_* d_1$ $c^*, c_* > 0$)

then A is bounded w.r.t. d_1 iff A is bounded w.r.t. d_2

(i.e. d_1, d_2 completely "agree" w.r.t. boundness)

E.g. $X = \mathbb{R}^d$, d_I, d_{max}, d_{II} and we know that every possible pair of them satisfies a relation of the previous type (*). Hence d_I, d_{max}, d_{II} completely "agree" w.r.t. boundness on $\mathbb{R}^d$.

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2. Boundness and Metric Subspaces

Remember: i. (X, d) metric space, $A \subseteq X$ ($\neq \emptyset$) we can "restrict" d to A (obtaining d_A) hence we obtain (A, d_A) as a metric subspace of the previous.

ii. $x \in A$, $\varepsilon > 0$, $O_{d_A}(x,\varepsilon) = O_d(x,\varepsilon) \cap A$

(something similar holds for closed balls)

iii. A is bounded iff $\exists\, x \in A, \varepsilon > 0$: $O_d(x,\varepsilon) \supseteq A$.

Lemma. (A, d_A) is bounded iff A is bounded w.r.t. d.

(If true it means that we may only consider the cases of whether a metric (sub-) space if bounded.)

Proof.

a. if A, d_A is bounded then A is a bounded subset of X.

b. if A is a bounded subset of X then A, d_A is bounded.

a. we know that A, d_A is bounded $\iff \exists\, x \in A, \varepsilon > 0$: $O_{d_A}(x,\varepsilon) \supseteq A$, but we have that $O_{d_A}(x,\varepsilon) \overset{\subseteq A}{=} O_d(x,\varepsilon) \cap A \Rightarrow O_d(x,\varepsilon) \supseteq A \iff A$ is a bounded subset of X. | hence $O_{d_A}(x,\varepsilon) \supseteq A \iff O_d(x,\varepsilon) \cap A \supseteq A \Rightarrow O_d(x,\varepsilon) \supseteq A$

b. we know that A is a bounded subset of $X \iff \exists\, x \in A, \varepsilon > 0$: $O_d(x,\varepsilon) \supseteq A$. Consider $O_{d_A}(x,\varepsilon) = O_d(x,\varepsilon) \cap A \supseteq A$. Hence A, d_A is bounded. $\square$

Useful Appendix:

d_A : the restriction of d on $A \times A$. If $x \in A$, $\varepsilon > 0$

$O_{d_A}(x,\varepsilon) = \{y \in A : d_A(x,y) < \varepsilon\} = \{y \in X : d(x,y) < \varepsilon\} \cap A = O_d(x,\varepsilon) \cap A$

$O_{d_A}[x,\varepsilon] = \dots = O_d[x,\varepsilon] \cap A$ |

$O_{d_A}[0_{2\times1}, 1]$ $x = 0_{2\times1}$, $\varepsilon = 1$

$\mathbb{R}^2, d_I$, $A = \{(x_1, x_2) : x_1, x_2 \geq 0\}$

3. "Fragility of boundness w.r.t. limit"

Consider $X = \mathbb{R}$, for $n = 1, 2, 3, \dots$ consider the
$\in \mathbb{N}^*$

following $d_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by:

$$x, y \in \mathbb{R}, \quad d_n(x,y) := \begin{cases} |x-y|, & |x-y| < n \\ n, & |x-y| \geq n \end{cases} = \begin{cases} |x-y|, & d_u(x,y) < n \\ n, & d_u(x,y) \geq n \end{cases}$$

$$= \min(d_u(x,y), n) = \begin{cases} d_u(x,y), & d_u(x,y) < n \\ n, & d_u(x,y) \geq n \end{cases}$$

Exer. Show that d_n is a well defined metric for any $n = 1, 2, \dots$

(If $n = 0$ were allowed, d_0 would be a pseudo-metric)

Hence we have essentially constructed a "sequence" of metric spaces $\big((\mathbb{R}, d_1), (\mathbb{R}, d_2), (\mathbb{R}, d_3), \dots, (\mathbb{R}, d_n), \dots\big)$

bounded

a. Consider $(\mathbb{R}, d_n)$, let $x \in \mathbb{R}$, and for $\varepsilon = n+1$

Consider furthermore

we have that $\max\limits_{z,y} d_u(z,y) = n$

$$O_{d_n}(x, n+1) = \{y \in \mathbb{R} : d_n(x,y) < n+1\} = \mathbb{R}.$$

Hence $\mathbb{R}$ is bounded w.r.t. d_n $\forall n = 1, 2, \dots$

b. What happens to $d_n(x,y)$ as $n \to +\infty$?

fixed

We have that eventually (for large enough n)

$d_n(x,y) = |x-y|$, hence $\lim_{n\to+\infty} d_n(x,y) = |x-y| = d_u(x,y)$

Since x,y were arbitrary we have that

$$\forall x,y \in \mathbb{R} \quad d_n(x,y) \xrightarrow[n\to+\infty]{} d_u(x,y)$$

↳ Pointwise Convergence (to be examined) ✓

(in some way - compare with the notion of pointwise convergence - d_n converges to d_u)

Hence in some way $\mathbb{R}, d_u$ is some kind of limit of the previous sequence of metric space.

c. $\mathbb{R}, d_u$ is <u>not</u> bounded

Hence we have an example for which boundness is "lost in the limit".

Function Spaces (again)

4. We have in many cases worked with $B(X, \mathbb{R}), d_{sup}$

$\{f: X \to \mathbb{R},\ \text{bounded}\}$

- f is bounded $\Leftrightarrow$ $f(X)$ is a bounded subset of $\mathbb{R}$ w.r.t d_u.
- $d_{sup}(f,g) = \sup_{x\in X} |f(x)-g(x)| = \sup_{x\in X} d_u(f(x), g(x))$.

We can now easily generalize the previous construction

Suppose that we have a metric space (X, d)

– $f: Y \to X$ ($Y \neq \emptyset$) will be considered bounded w.r.t. d

iff $f(Y)$ is a bounded subset of X w.r.t. d

i.e. $\exists z \in X, \varepsilon > 0$: $O_d(z, \varepsilon) \supseteq f(Y)$, hence

$B(Y, X)$ is well defined. (always $\neq \emptyset$ – constant functions) $\neq \emptyset$ ✓

– We can analogousle define the analogue of d_{sup} in this extended setting:

$$f, g \in B(Y,X), \quad d^*_{sup}(f,g) = \sup_{x \in Y} d(f(x), g(x))$$

$$\boxed{d_{sup}(f,g) = \sup_{x \in Y} |f(x) - f(y)| = \sup_{x \in Y} d_u(f(x), f(y))}$$

Exe. Show that d^*_{sup} is a well-defined metric.

[$\Rightarrow$ When $X = \mathbb{R}$, $d = d_u$ then we recover our "usual category of examples]

An Equivalent Definition (useful)

5. (X, d), $\varepsilon > 0$, $Z \subseteq X$. Consider the following collection of open balls:

$$\underbrace{\mathcal{O}(Z, \varepsilon)}_{\text{a set of balls of a given } \varepsilon} = \{ O_d(x, \varepsilon), \ \forall x \in Z \}$$

Parameterized by the centers in Z

The relevant collection of closed balls is

$$\mathcal{O}(Z, \varepsilon) = \{ O_d[x, \varepsilon], \ \forall x \in Z \}$$

• When Z is finite then the collections above are termed finite.

Definition:
When $A \subseteq X$, we say that A is covered by $\mathcal{O}(Z,\varepsilon)$ iff $\bigcup_{x \in Z} O_d(x,\varepsilon) \supseteq A$

(analogously for $\bar{\mathcal{O}}(Z,\varepsilon)$).

An equivalent def. will lead us to total boundedness

(Another characterization of boundedness)

Theorem. A is bounded w.r.t. d iff $\exists\, \varepsilon > 0$, w.r.t. which there exists a finite $\mathcal{O}(Z,\varepsilon)$ that covers A.

[we can equivalently use closed balls - Exe!]

Proof.

a. Suppose that A is bounded $\Leftrightarrow \exists\, x, \varepsilon > 0$: $O_d(x,\varepsilon) \supseteq A$. Hence $\mathcal{O}(Z,\varepsilon) = \{O_d(x,\varepsilon)\}$, $Z = \{x\}$.

b. Suppose that there exists $\varepsilon > 0$ w.r.t. there exists a finite $\mathcal{O}(Z,\varepsilon)$ that covers A.

$\mathcal{O}(Z,\varepsilon)$ has the form $\{O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \ldots, O_d(x_n,\varepsilon)\}$ for some $n \in \mathbb{N}^*$. ($Z = \{x_1, x_2, \ldots, x_n\}$)

Construct a single encompassing ball

We will try to construct a "large enough" ball that covers $\bigcup_{i=1}^{n} O_d(x_i,\varepsilon)$. Without loss of generality choose x_1 as center

Let $\delta := \max\limits_{i=2,\dots,n} d(x_1, x_i) + \varepsilon + 1 > 0$

this is a finite set of distances

this is well defined

$\forall y \in \bigcup\limits_{i=1}^{n} O_d(x_i, \varepsilon) \Rightarrow y \in O_d(x_1, \delta)$

We have that $O_d(x_1, \delta) \supseteq \bigcup\limits_{i=1}^{n} O_d(x_i, \varepsilon)$, this is due to that: let $y \in \bigcup\limits_{i=1}^{n} O_d(x_i, \varepsilon) \Rightarrow y \in O_d(x_i, \varepsilon)$ for some i, that is $d(x_i, y) < \varepsilon$ for some i.

We have that $d(x_1, y) \leq d(x_1, x_i) + d(x_i, y)^{<\varepsilon}$

$$< \max_{i=2,\dots,n} d(x_1, x_i) + \varepsilon$$

$$< \max_{i=2,\dots,n} d(x_1, x_i) + \varepsilon + 1 = \delta$$

$\leq \max\limits_{i=2,\dots,n} d(x_1, x_i)$

Hence $d(x_1, y) < \delta \iff y \in O_d(x_1, \delta) \Rightarrow$

$$O_d(x_1, \delta) \supseteq \bigcup_{i=1}^{n} O_d(x_i, \varepsilon) \supseteq A$$

hence $A \subseteq O_d(x_1, \delta)$ hence A is bounded. $\square$

End of lecture 5.

Total Boundness

Definition. A is totally bounded (w.r.t. d) iff $\forall \varepsilon > 0$ $\exists Z$ finite such that the collection $\mathcal{O}(Z, \varepsilon)$ covers A.

[Dually we can consider collections of closed balls].

Remark. The definition allows the dependence of Z on ε. Hence both the centers and the cardinality of Z is allowed to change with ε.

Remark. When A is totally bounded then $\forall \varepsilon > 0$ Z can be chosen as a subset of A (i.e. the ball centers can be chosen to lie inside A).

Proof. Suppose that A is totally bounded. Let $\varepsilon > 0$. Due to the previous for $\delta = \varepsilon/2$ there exists a finite Z such that the collection $\mathcal{O}(Z, \varepsilon/2)$ covers A. The collection $\mathcal{O}(Z, \varepsilon/2)$ is the set of open balls $\{O_d(x_1, \varepsilon/2), O_d(x_2, \varepsilon/2), \dots, O_d(x_n, \varepsilon/2)\}$ ($Z = \{x_1, x_2, \dots, x_n\}$ for n that may depend on $\varepsilon/2$).

If $Z \subseteq A$ then the collection $\mathcal{O}(Z, \varepsilon)$ covers A since $O_d(x_i, \varepsilon/2) \subseteq O_d(x_i, \varepsilon)$ $\forall i = 1, \dots, n$. If $Z \nsubseteq A$ then:

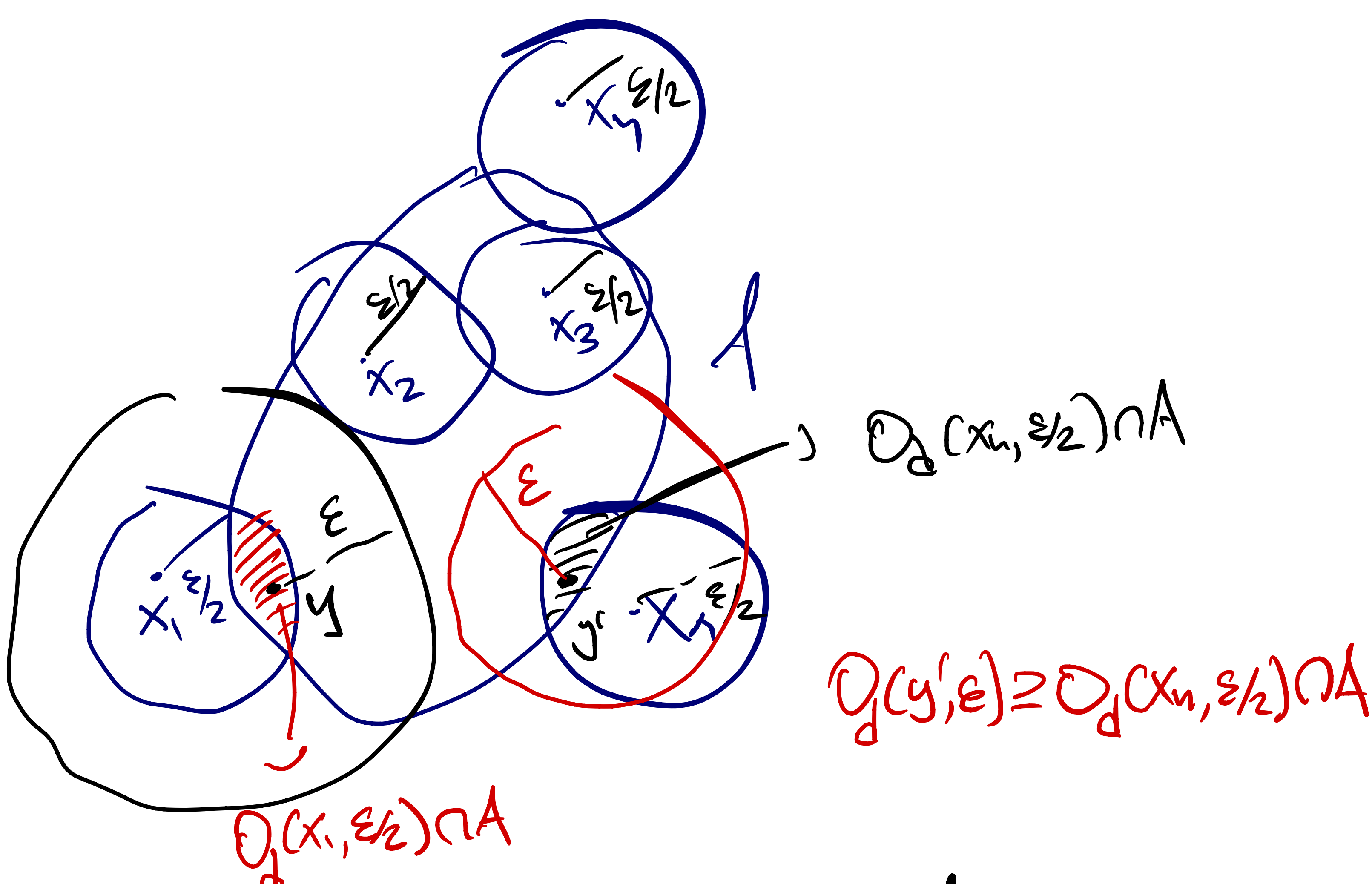
Suppose that $x_1 \notin A$ without loss of generality.

Since $\bigcup_{i=1}^{n} O_d(x_i, \varepsilon/2) \supseteq A$ we consider $O_d(x_1, \varepsilon/2) \cap A$. If this is empty then the ball $O_d(x_1, \varepsilon/2)$ is not needed in the covering and thereby x_1 can be discarded from Z and we can move on to the next element of Z that does not lie in A.

So suppose that $O_d(x_1, \varepsilon/2) \cap A$ is not empty.

Let $y \in O_d(x_1, \varepsilon/2)$. Consider $O_d(y, \varepsilon)$. We have that if $z \in O_d(x_1, \varepsilon/2) \cap A$ then $z \in O_d(y, \varepsilon)$.

[redacted]



$$O_d(y, \varepsilon) \supseteq O_d(x_1, \varepsilon/2) \cap A$$

[redacted]

(Continue the proof of that when A is t.b., the ball centers of the collections that cover it can be chosen to lie inside A, $\forall \varepsilon>0$).

Reminder: For $\varepsilon>0$, we considered $\mathcal{O}_{\varepsilon/2}(Z)$ a finite cover of open balls of radius $\varepsilon/2$ that covers A.

$Z=\{x_1, x_2, \dots, x_n\}$. We supposed that $x_1 \notin A$.

We considered the case $O_d(x_1, \varepsilon/2) \cap A$.

We have chosen $y \in O_d(x_1, \varepsilon/2) \cap A$, and considered the ball $O_d(y, \varepsilon)$: we have that $O_d(y,\varepsilon) \supseteq O_d(x_1, \varepsilon/2) \cap A$.

This is due to that: if $z \in O_d(x_1, \varepsilon/2) \cap A$, then

$$d(y,z) \le \underbrace{d(x_1,y)}_{<\varepsilon/2} + \underbrace{d(x_1,z)}_{<\varepsilon/2} < \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow$$

z arbitrary

$$d(y,z) < \varepsilon \iff z \in O_d(y,\varepsilon) \Rightarrow O_d(y,\varepsilon) \supseteq O_d(x_1, \varepsilon/2) \cap A$$

Obviously we can repeat the previous procedure for any $x_i \in Z$ such that $x_i \notin A$.

Hence for any $x_i \in Z$ and $x_i \notin A$, and $O_d(x_i, \varepsilon/2) \cap A \neq \emptyset$, we can find some $y_i \in A$ such that $O_d(y_i,\varepsilon) \supseteq O_d(x_i, \varepsilon/2) \cap A$.

For each $x_i \in Z$, for which $x_i \in A$ we consider $O_d(x_i, \varepsilon)$.

Obviously due to monotonicity we have that $O_d(x_i,\varepsilon) \supseteq O_d(x_i,\varepsilon/2) \cap A$. Hence we have constructed a finite collection of open balls or radius ε with: i. centers inside A and ii. the collection covers A. The result follows since ε is arbitrary. $\square$

Lemma. The previous hold if instead of covers by open balls we had considered covers of closed balls.

Proof. (covers of open balls $\Rightarrow$ covers of closed balls)
Suppose A is t.b. (based on covers of open balls) and $\varepsilon > 0$. There exists a cover of open balls, say,

$$O_\varepsilon(Z) = \{ O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \ldots, O_d(x_n,\varepsilon) \}$$

such that $\bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq A$. We have that $O_d(x_i,\varepsilon) \subseteq O_d[x_i,\varepsilon]$ $\forall i = 1,2,\ldots,n \Rightarrow \bigcup_{i=1}^{n} O_d[x_i,\varepsilon] \supseteq A$. Hence

$\{ O_d[x_1,\varepsilon], O_d[x_2,\varepsilon], \ldots, O_d[x_n,\varepsilon] \}$ is the required collection. (Hence the definition of t.b. using open balls implies the definition of t.b. using closed balls)

(covers of closed balls $\Rightarrow$ covers of open balls)

Suppose that A satisfies the definition of t.b. w.r.t. covers of closed balls. Let $\varepsilon>0$: we know that there exists a finite collection of closed balls

$$\{ O_d[x_1,\varepsilon/2], O_d[x_2,\varepsilon/2], \dots, O_d[x_n,\varepsilon/2] \}$$ that covers A. We know that $O_d(x_i,\varepsilon) \supseteq O_d[x_i,\varepsilon/2]$ $\forall i=1,\dots,n$

$$\Rightarrow \bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq \bigcup_{i=1}^{n} O_d[x_i,\varepsilon/2] \supseteq A$$ hence the collection $\{ O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \dots, O_d(x_n,\varepsilon) \}$ is the required one. Since ε is arbitrary the result follows. $\square$

Further useful results:

- X,d will be considered totally bounded iff X is totally bounded w.r.t. as a subset of itself.
- $A \subseteq X$ is totally bounded w.r.t. d, iff A, d_A is totally bounded (Proof - Exercise! Use the result on the choice of the ball centers - it is similar to the proof of the analogous result for boundness).

Lemma If A is totally bounded w.r.t. d then it is also bounded w.r.t. d.

Proof. [Check the definitions using the covers!]

Lemma. If A is finite then it is totally bounded. (this holds for any metric!)

Proof. If $A = \emptyset$ then A is trivially t.b. Suppose that $A \neq \emptyset$, i.e. it is of the form

$$A = \{x_1, x_2, \dots, x_n\}, \quad x_i \in X \ \forall i = 1, \dots, n$$

For $\varepsilon > 0$, consider $\{O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \dots, O_d(x_n,\varepsilon)\}$

Since $x_i \in O_d(x_i,\varepsilon)$ $\forall i = 1, \dots, n$, we have that

$$\bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq A.$$ Since ε is arbitrary the result follows. $\square$

Lemma If A is t.b. w.r.t. d and $B \subseteq A$ then B is also t.b. w.r.t. d.

Proof. $\forall \varepsilon > 0$ the cover of open balls covers A it necessarily covers B.

Dually: if A is not t.b. w.r.t. d and $B \supseteq A$ then also B is not t.b. w.r.t. d

(A is not t.b. iff $\exists \varepsilon > 0$: every finite collection of open balls of radius ε cannot cover A).

Examples and Counter-examples

1. X, d_d (X, d_d is always bounded)

We will show that: X, d_d is totally bounded iff it is finite.

Proof a. (Finite $\Rightarrow$ t.b) (see above).

b. (if X, d_d is t.b. then it is finite).

Suppose that X is infinite. Let $\varepsilon = 1/2$.

(Reminder $O_{d_d}(x, 1/2) = \{x\}$ $\forall x \in X$). This directly implies that the only way to cover X by open balls of radius $1/2$ is: $\bigcup_{x \in X} O_{d_d}(x, 1/2)$

$$\bigcup_{x \in X} O_{d_d}(x, 1/2) = \bigcup_{x \in X} \{x\} = X$$

but the collection $\{O_{d_d}(x, 1/2), x \in X\}$ is not finite since X is infinite. The result follows. $\square$

Hence the previous implies via example that total boundness is strictly stronger than boundness.

2. Is $\mathbb{R}, d_u$ totally bounded? No since it is not even bounded. The same holds for $\mathbb{R}^d, d_I$.

(the interesting question here is when $A \subseteq \mathbb{R}^d$ is t.b. w.r.t. d_I).

3. Remember $B(\mathbb{N}, \mathbb{R}), d_{sup}$ (bounded real sequences with the uniform metric).

$$A = \left\{ f_m \in B(\mathbb{N}, \mathbb{R}),\ f_m(n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases},\ m \in \mathbb{N} \right\}$$

Reminder: infinite dimensional vector representation of f_m

$$(0, 0, \ldots, 1, 0, \ldots 0, \ldots)$$

$(m+1)^{th}$ position

e.g. f_0 $\quad (1, 0, 0, \ldots, 0, \ldots)$

We have proven that A is bounded w.r.t.

d_{sup} (we have used uniform boundness).

Is A totally bounded w.r.t. d_{sup}? A is not totally bounded due to that: we know that if A were t.b. then the centers of the covering balls can be chosen to belong to A.
Also we have that $d_{sup}(f_n, f_m) = \begin{cases} 0, & n=m \\ 1, & n\neq m. \end{cases}$
A is obviously infinite. For $\varepsilon = 1/2$

$O_{d_{sup}}(f_m, 1/2) \cap A = \{f_m\}$. Hence in order to be able to cover A with open balls of the form $O_{d_{sup}}(f_m, 1/2)$, $m \in \mathbb{N}$ we need as many of those balls as there are elements of.
But A is infinite. Hence A is not totally bounded w.r.t. d_{sup}. $\square$

In order to construct examples the following notion will be usefull:

Covering Numbers and Metric Entropy

We will use without loss of generality covers by closed balls.

Framework: X, d , $A \subseteq X$, $\varepsilon > 0$

Definition: The covering number of A w.r.t. ε and d, $N(\varepsilon, d, A) :=$ Min # of closed balls of radius ε that cover A.

Remark: it is possible to prove that $N(\varepsilon, d, A)$ is unique $\forall \varepsilon, d, A$.

$\ln N(\varepsilon, d, A)$ is the metric entropy number w.r.t. ε and d.

- It is easy to show that A is t.b. w.r.t. d iff $N(\varepsilon, d, A) \in \mathbb{N}$ (i.e. it is $< +\infty$) $\forall \varepsilon > 0$.

  (dually A is not t.b. w.r.t. d iff $\exists \varepsilon > 0$: $N(\varepsilon, d, A) = +\infty$)
- It is easy to see that $N(\varepsilon, d, A)$ is decreasing

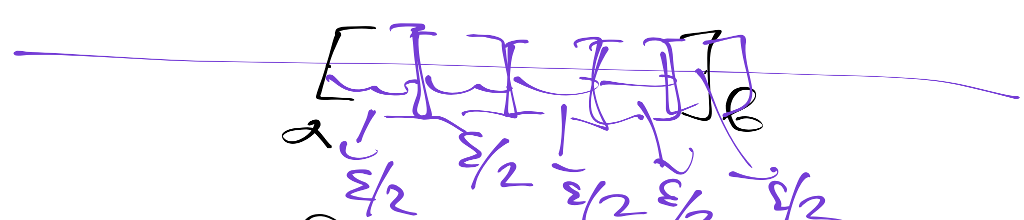
w.r.t. ε for fixed d, A.

- <u>Generally</u>, as $\varepsilon \downarrow 0$ $\left(\begin{smallmatrix}\varepsilon \to 0 \\ \varepsilon > 0\end{smallmatrix}\right)$, $N(\varepsilon, d, A) \to +\infty$ for fixed d, A

  Even when A is t.b.

- What is very informative on the complexity of A as a metric (sub-)space is the rate at which $N(\varepsilon, d, A)$ (or equivalently $\ln N(\varepsilon, d, A)$) diverges as $\varepsilon \downarrow 0$.

E.g. $X = \mathbb{R}$, $d = d_u$ $\qquad [\alpha, \beta] = O_{d_u}\left[\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}\right]$



$$N(\varepsilon, d_u, [\alpha, \beta]) = \begin{cases} 1, & \varepsilon \geq \frac{\beta-\alpha}{2} \\ \text{ο μικρότερος φυσικός} \geq \frac{\beta-\alpha}{2\varepsilon}, & \varepsilon < \frac{\beta-\alpha}{2} \end{cases} < +\infty$$

$\forall \varepsilon > 0$. Hence $[\alpha, \beta]$ is t.b. w.r.t. d_u

Hence we have shown that $\forall x \in \mathbb{R}, \varepsilon > 0$, $O_{d_u}[x, \varepsilon]$ is totally bounded w.r.t. d_u. Hence if $A \subseteq \mathbb{R}$ is bounded w.r.t. d_u then it is also t.b. w.r.t. d_u.

Hence Boundness $\Leftrightarrow$ Total Boundness inside $\mathbb{R}, d_u$