# Lecture 4

Balls and Relations

Second Countability

Boundedness

Given an arbitrary Metric space $(X,d)$,

$x \in X$, $\varepsilon > 0$ :

Open Ball centered at $x$ of radius $\varepsilon$,

$$O_d(x,\varepsilon) = \{ y \in X : d(x,y) < \varepsilon \}$$

Closed Ball centered at $x$ of radius $\varepsilon$,

$$O_d[x,\varepsilon] = \{ y \in X : d(x,y) \leq \varepsilon \}$$

* Never empty (though in some case "peculiar"), obey inclusion relations for fixed $x$.

* Seperate points ($\Rightarrow$ uniqueness of limits)

$C_{I}$. Balls and dominance

Consider $(X,d_1)$, $(X,d_2)$, $x \in X$, $\varepsilon > 0$ and $O_{d_1}(x,\varepsilon)$, $O_{d_2}(x,\varepsilon)$. Suppose that $\exists c > 0$:

$d_1 \leq c\, d_2$ [functional inequality]

Let $y \in O_{d_2}(x,\varepsilon) \Leftrightarrow d_2(x,y) < \varepsilon \overset{c>0}{\Leftrightarrow} c\,d_2(x,y) < c\varepsilon$ $\overset{d_1(x,y)}{\Rightarrow} d_1(x,y) < c\varepsilon$

done. $\Rightarrow$ $y \in O_{d_1}(x,c\varepsilon) \Rightarrow O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x,c\varepsilon)$

Hence $\forall x \in X, \varepsilon > 0$, $\left( d_1 \leq c\, d_2 \Rightarrow \boxed{O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x,c\varepsilon)} \right)$ (*)

antitone transformation. ☑

- Notice that (*) is equivalent to:

$$\exists c>0 : d_1 \leq c\, d_2 \Rightarrow \forall \delta>0, \forall x\in X$$

$$O_{d_2}(x, \delta/c) \subseteq O_{d_1}(x,\delta)$$

simply set in the previous $\varepsilon = \delta/c$ and notice that since $c>0$ the transformation $\delta \to \delta/c$ is bijective. $[(0,+\infty) \to (0,+\infty)]$

- Furthermore, if $\exists\, c, c^* > 0 : c^* d_2 \leq d_1 \leq c\, d_2$ Equivalence

then $\forall x\in X$, $\forall \delta, \varepsilon > 0$

$$\begin{cases} O_{d_2}(x, \delta/c) \subseteq O_{d_1}(x,\delta) \\ O_{d_1}(x, c^*\varepsilon) \subseteq O_{d_2}(x,\varepsilon) \end{cases}$$

and setting $\varepsilon = \delta/c$ we obtain

$$\forall x\in X, \varepsilon>0 : \boxed{O_{d_1}(x, c^*\varepsilon) \subseteq O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x, c\varepsilon)}$$

& Show that the above also hold for closed balls.

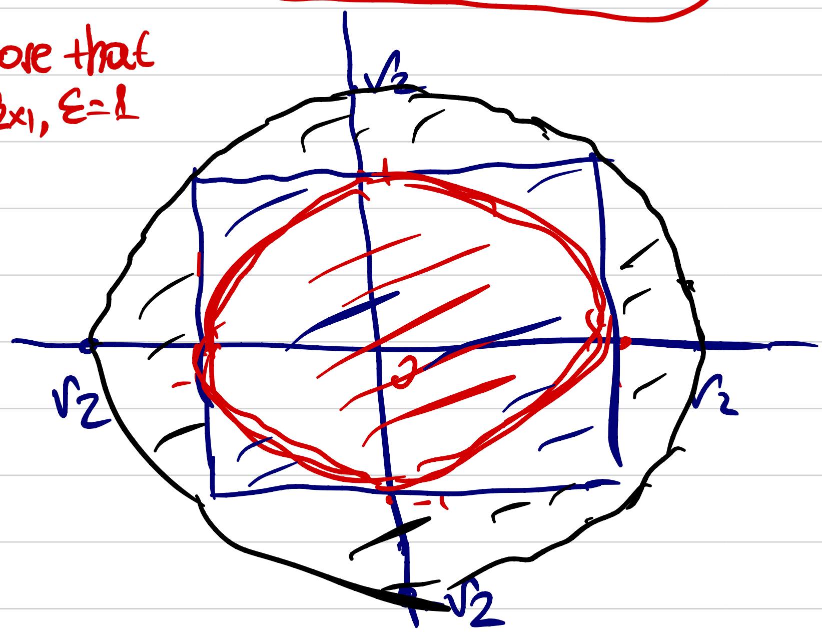
Hence functional inequality relations between metrics imply inclusion relations between corresponding balls of the same center.

A simple geometric example:

For $X=\mathbb{R}^2$ we already know (see lecture 2)

$$\boxed{d_{max} \leq d_I^{\,d_1} \leq \sqrt{2}\, d_{max}^{\,d_2}} \quad (c^*=1, c=\sqrt{2})$$

And that suppose that $x=\mathbb{O}_{2\times1}, \varepsilon=1$




$$O_{d_I}[\mathbb{O}_{2\times1},1] \subseteq O_{d_{max}}[\mathbb{O}_{2\times1},1] \subseteq O_{d_I}[\mathbb{O}_{2\times1},\sqrt{2}]$$

$$\longleftrightarrow$$

Ex. Depict the analogous geometry for every remaining pair of equivalencies between $d_I, d_{max}, d_{II}$.

$C_{III}$. Neighborhoods of Balls and local countable information

For $x \in X$ consider the collection:

$$\{ O_d(x,\varepsilon), \ \varepsilon > 0 \}$$ [dually $\{O_d[x,\varepsilon], \ \varepsilon>0\}$]

E.g. in a discrete space this is finite

$$\{ \{x\}, X \}$$

but in $\mathbb{R}, d_u$ it is uncountable

$$\{ (x-\varepsilon, x+\varepsilon), \ \varepsilon>0 \}$$ ⇒ conveys local information

is a neighborhood system of balls centered at $x$

In any case the sub-collection (always countable)

$$\{ O_d(x, 1/n), \ n \in \mathbb{N}^* \}$$ [dually $\{O_d[x, 1/n], \ n \in \mathbb{N}^*\}$]

retains in some sense the same information since

$$\forall \varepsilon>0, \ \exists n^*(\varepsilon) \in \mathbb{N}^* : O_d(x, 1/n^*(\varepsilon)) \subseteq O_d(x,\varepsilon)$$

: for $\varepsilon>0$ see $n^*(\varepsilon)$ the smallest positive natural greater than or equal to $1/\varepsilon$. $\Rightarrow \frac{1}{n^*(\varepsilon)} \leq \varepsilon$

Hence the local information included in the neighborhood systems (to be made precise later in the course) is reducible to countable sub-collections [Second Countability of Metric Spaces] ☒

We are now ready to discuss potential properties of Metric Spaces:

1. $\boxed{\text{Boundedness}}$

Remember (Lecture 1) that $A \subseteq \mathbb{R}$ is bounded iff it has an infimum and a supremum $\Leftrightarrow$ $A \subseteq [\inf A, \sup A] \Leftrightarrow \exists x \in \mathbb{R}, \varepsilon > 0 \;\; A \subseteq (x-\varepsilon, x+\varepsilon)$

i.e. if $A$ can be covered by an open (eq. closed) interval of finite radius (i.e. an open (closed) ball in the usual metric).

Hence the notion is generalisable in every Metric space:

Definition. $(X,d)$ is a Metric space and $A \subseteq X$. $A$ is called bounded (w.r.t. $d$ -d-bounded) iff $\exists x \in X, \varepsilon > 0 : A \subseteq O_d(x,\varepsilon)$. ∎

does not necessarily belongs to $A$

existential quantifier

*a. An equivalent definition can be expressed via closed balls. If $\exists x \in X, \varepsilon > 0 : A \subseteq O_d(x,\varepsilon)$ the $A \subseteq O_d[x,\varepsilon]$ due to inclusion. If $\exists x \in X, \delta > 0 : A \subseteq O_d[x,\delta]$ then $A \subseteq O_d(x,\delta+1)$ due to inclusion. ∎

$\exists x \in X, \varepsilon > 0 : A \subseteq O_d[x,\varepsilon]$

$\subseteq O_d[x,\varepsilon]$

$\subseteq O_d(x,\delta+1)$

(remember the ball inclusion properties)

*b. The center of the covering ball need not reside in $A$. However (when $A \neq \emptyset$) $A$ is bounded $\iff \exists y \in A, \varepsilon > 0 : A \subseteq O_d(y,\varepsilon)$.

Proof. $(\Leftarrow)$ obvious from the definition

$(\Rightarrow)$ if $\exists x \in X, \varepsilon > 0 : A \subseteq O_d(x,\varepsilon)$ choose $y \in A$ and consider $O_d(y, \varepsilon + d(x,y))$.

If $y = x$ (hence $x \in A$) $\Rightarrow A \subseteq O_d(y,\varepsilon)$.

If $y \neq x \Leftrightarrow d(x,y) > 0$ and if $z \in A$ then

$$d(y,z) \leq d(y,x) + \underbrace{d(x,z)}_{<\varepsilon} < d(y,x) + \varepsilon$$

tr. ineq. for d

since $z \in A$ $\Rightarrow z \in O_d(x,\varepsilon)$

radius of the new ball

$\Leftrightarrow z \in O_d(y, d(y,x)+\varepsilon) \Leftrightarrow A \subseteq O_d(y, d(y,x)+\varepsilon)$.

$\boxminus$

Hence if $A$ is bounded the center of the covering ball can be chosen to lie in $A$ (possibly adjusting radius)

$\forall d$ (universally)

*c. $\emptyset$ is always bounded. Every open or closed ball is d-bounded (obvious) *d. $(X,d)$ is bounded iff $X$ is a $(d\text{-})$ bounded subset of itself.

*e. If $B \subseteq A$ and $A$ $(d\text{-})$ bounded then $B$ is also $(d\text{-})$ bounded $(B \subseteq A \subseteq O_d(x,\varepsilon)$ — Hereditarity)

*f. (Deny the definition) $A$ is not $(d\text{-})$ bounded iff $\forall x \in X, \varepsilon > 0 : A \nsubseteq O_d(x,\varepsilon) \iff$ ✓

$\forall x \in X, \varepsilon > 0, \exists y \in A : y \notin O_d(x,\varepsilon)$

transformed to the universal quantifier

e.g. $(\mathbb{R}, d_u)$ is not bounded since $\forall x \in \mathbb{R}, \varepsilon > 0$

$y := x + 2\varepsilon \notin (x-\varepsilon, x+\varepsilon)$. ✓

* $\forall x \in X, \varepsilon > 0$, $O_d(x,\varepsilon)$, $O_d[x,\varepsilon]$ are d-bounded since they are "self-covered"

Eg. (Metrics Comparison) Suppose that $\exists c>0$:

$d_1 \leq c\, d_2$ and $A$ is $d_2$-bounded.

Then $A$ is also $d_1$-bounded:

We know that $\forall x \in X,\ \varepsilon>0$, $O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x, c\varepsilon)$

Since $A$ is $d_2$-bounded, $\exists x \in X,\ \varepsilon>0$:

$A \subseteq O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x,c\varepsilon)$, hence $A$ is $d_1$-bounded.

[Boundedness w.r.t. the dominant Metric $\Rightarrow$ boundedness w.r.t. the dominated Metric]

If Moreover $\exists c^*, c>0$: $\overset{\text{equivalence}}{c^* d_2 \leq d_1 \leq c\, d_2}$

then obviously from the above:

Boundedness w.r.t. $d_1$ $\Leftrightarrow$ Boundedness w.r.t. $d_2$

e.g. In $\mathbb{R}^n$ $A$ is $d_*$-bounded iff it is $d_{**}$-bounded

where $*, ** = I, \max, \| \|$ $\square$

This is a first example of "properties similarity": two equivalent metrics identify the same subsets of X as bounded (it could be useful: eg. Maybe easier to show that A is $d_\infty$-bounded)

* If A is finite $\Rightarrow$ it is d-bounded $\forall d$ (universally)
Suppose $A \neq \emptyset$. Set $\delta := \max\limits_{x,y \in A} d(x,y)$. ✓ Since A finite $0 < \delta < +\infty$
then $A \subseteq O_d(x, \delta+1)$ $\forall x \in A$, since if $y \in A$, $d(x,y) \leq \max\limits_{z,h \in A} d(z,h) < \delta + 1$
$\Rightarrow y \in O_d(x, \delta+1)$
$\Rightarrow A \subseteq O_d(x, \delta+1)$ ∎

Examples

1. Every discrete space is bounded:

$(X, d_1)$, $\forall x \in X$ and $\varepsilon > 1$ $O_{d_1}(x, \varepsilon) = X \supseteq X$.

(another peculiarity for discrete spaces; it also shows that boundedness depends on the metric: $\mathbb{R}, d_1$ is bounded but $\mathbb{R}, d_u$ is not.

2. $\mathbb{R}^n$, $d_{max}$ is not bounded. Consider $x \in \mathbb{R}^n$, $\varepsilon > 0$ and $O_{d_{max}}(x, \varepsilon)$. Let

$[x = (x_1, x_2, \dots, x_n)]$

$$y := \begin{pmatrix} x_1 + 2\varepsilon \\ x_2 + 2\varepsilon \\ \vdots \\ x_n + 2\varepsilon \end{pmatrix}$$ and notice that

$$d_{max}(x,y) = \max_{1 \le i \le n} |x_i - y_i| = \max_{1 \le i \le n} |2\varepsilon| = 2\varepsilon > \varepsilon$$

$\Rightarrow y \notin O_{d_{max}}(x, \varepsilon)$. ▯

3. $\mathbb{R}^n, d_I$ and $\mathbb{R}^n, d_{II}$ are not bounded

Follow directly from 2 and $x_g$ and equivalence. ▯

4. Consider $\mathbb{R}^n, d_{II}$ and let $A = \{x \in \mathbb{R}^n : x_i = 0 \text{ or } 1, i = 1, \dots, n\}$ [remember $A, d_I$ is the Hamming Space]

Then $O_{d_{II}}[0_{n\times1}, n]$ covers $A$ since if $x \in A$:

$$d_{II}(\underset{\in A}{0_{n\times1}}, \underset{\in A}{x}) = \#\{i = 1, \dots, n : \underset{x_i = 1}{x_i \neq 0}\} \le n$$

$\Rightarrow x \in O_{d_{II}}[0_{n\times1}, n]$.

Hence the Hamming space is bounded. ☒

5. For $Y \neq \emptyset$ consider $(B(Y,\mathbb{R}), d_{sup})$.

Let $A \subseteq B(Y,\mathbb{R})$

Definition. $A$ is uniformly bounded iff

$$\sup_{f \in A} \left( \sup_{x \in Y} |f(x)| \right) < +\infty$$

→ this holds for every $f$

→ this is stronger; it requires a joint property for the $A$ family.

Important

Lemma. $A \neq \emptyset$ is ($d_{sup}$-) bounded $\iff$ $A$ is uniformly bounded.

Proof. ($\Leftarrow$) Suppose that $\sup_{f \in A} \sup_{x \in Y} |f(x)| < +\infty$

Set $\delta := \sup_{f\in A} \sup_{x\in Y} |f(x)|$. If $\delta = 0 \iff$

$f(x) = 0 \quad \forall x \in Y, \forall f \in A \iff A := \{\mathbf{0}\}$

(where $\mathbf{0} : Y \to \mathbb{R}$, $\mathbf{0}(x) = 0 \ \forall x \in Y$ - obviously $\mathbf{0} \in B(Y, \mathbb{R})$ - why?) hence $A$ bounded as finite.

If $\delta > 0$ then consider $O_{d_{\sup}}[\mathbf{0}, \delta]$.

Then if $f \in A$, $d_{\sup}(\mathbf{0}, f) = \sup_{x\in Y} |f(x) - \mathbf{0}(x)|$

$$= \sup_{x\in Y} |f(x)| \leq \sup_{g\in A} \sup_{x\in X} |g(x)| = \delta \Rightarrow f \in O_{d_{\sup}}[\mathbf{0}, \delta].$$

Hence $A$ is $d_{\sup}$-bounded.

($\Leftarrow$) Suppose now that $A$ is $d_{\sup}$-bounded. Hence

$\exists g \in A, \varepsilon > 0 : A \subseteq O_{d_{\sup}}[g, \varepsilon]$

But $\sup_{f\in A} \sup_{x\in Y} |f(x)| = \sup_{f\in A} \sup_{x\in Y} (|f(x) \pm g(x)|)$ ← to

$$= \sup_{f\in A} \sup_{x\in Y} |(f(x)-g(x))+g(x)| \le \sup_{f\in A}\sup_{x\in Y}\Big[|f(x)-g(x)|+ |g(x)|\Big] \le \underbrace{\sup_{f\in A}\sup_{x\in Y}|f(x)-g(x)|}_{f\in O_{d_{sup}}[g,\varepsilon]} + \underbrace{\sup_{f\in A}\sup_{x\in Y}|g(x)|}_{\text{independent of } f}$$

$g\in B(Y,\mathbb{R})$

$$= \sup_{f\in A} \underbrace{d_{sup}(f,g)}_{\le \varepsilon} + \underbrace{\sup_{x\in Y}|g(x)|}_{<+\infty} < +\infty, \text{ hence}$$

$$\le \sup_{f\in A}\varepsilon + \sup_{x\in Y}|g(x)| = \underbrace{\varepsilon}_{<+\infty} + \underbrace{\sup_{x\in Y}|g(x)|}_{<+\infty}$$

$A$ is uniformly bounded. ∎ End of lecture 4.

* Uniform boundedness: an analytically convenient way to characterize ($d_{sup}$-) boundedness.

Subexamples:

1. $Y=\mathbb{N}$, $B(\mathbb{N},\mathbb{R})$ space of bounded real sequences equipped with the uniform metric.

Let $A:\{e_i=(0,0,\dots,1,0,\dots 0,\dots),\ i\in\mathbb{N}\}$
↳ $i^{th}$ position

Then $\sup\limits_{f\in A}\sup\limits_{x\in Y}|f(x)|=\sup\limits_{i\in\mathbb{N}}\sup\limits_{n\in\mathbb{N}}|e_{in}|$

$$=\sup_{i\in\mathbb{N}}\sup_{n\in\mathbb{N}}\left|\begin{cases}0 & i\neq n\\ 1 & i=1\end{cases}\right| = \sup_{i\in\mathbb{N}}\sup_{n\in\mathbb{N}} 1 = 1 < +\infty$$

Hence $A$ uniformly bounded $\Leftrightarrow$ $d_{sup}$ bounded subset of $B(\mathbb{N},\mathbb{R})$. $\boxed{}$

2. $Y=[0,1]$, $B([0,1],\mathbb{R}), d_{sup}$

$L>0$, $A_L=\{f:[0,1]\to\mathbb{R},\ f(0)=0,\ \forall x,y\in[0,1],$
$|f(x)-f(y)|\leq L|x-y|\}$

We have proven that $\emptyset\neq A_L\subseteq B([0,1],\mathbb{R})$

Is it uniformly bounded?

$\sup\limits_{f\in A_L}\sup\limits_{x\in[0,1]}|f(x)|=\sup\limits_{f\in A_L}\sup\limits_{x\in[0,1]}|f(x)-f(0)|\leq$

$$\sup_{f \in A_L} \sup_{x \in [0,1]} \overbrace{\underbrace{L\,|x-0|}_{\text{independent of } f}}^{\geq 0} = L \sup_{x\in[0,1]} x = L \cdot 1 = L < +\infty .$$

Hence $A_L$ is uniformly bounded $\Leftrightarrow$ $d_{sup}$ - bounded subset of $B([0,1], \mathbb{R})$. $\blacksquare$