Lecture 13

Applications of BFPT

- Bellman Equation
- Picard-Lindelöf Theorem

Remember: [Blackwell's Lemma]

— $d = d_{sup}$

— $X \subseteq B(Z, \mathbb{R})$ contains every constant function.

Furthermore, if $f \in X$, $\alpha \in \mathbb{R}$ then $f + \alpha \in X$.

(α can be perceived as a constant function $Z \to \mathbb{R}$, $\alpha(x) = \alpha$ $\forall x \in Z$)

$(f+\alpha)(x) = f(x) + \alpha(x)$
$= f(x) + \alpha$

(e.g. $B(Z, \mathbb{R})$ satisfies the above)

— if $f, g \in X$ then $f \gtrsim g \Leftrightarrow f(x) \geq g(x)$ $\forall x \in Z$. (Definition)

(partial order on X)

Blackwell Hypotheses:

BL1. if $f \gtrsim g \Rightarrow \Phi(f) \gtrsim \Phi(g)$ (Monotonicity of Φ)

$\Updownarrow$

$\left[\begin{array}{l} \Phi(f), \Phi(g) : Z \to \mathbb{R} \in X \\ (\Phi(f))(x) \geq (\Phi(g))(x), \ \forall x \in Z \end{array} \right]$

BL2. $\exists \, \delta \in (0,1) : \forall f \in X, \alpha \in \mathbb{R}$

$$\Phi(f) + \delta\alpha \gtrsim \Phi(f + \alpha)$$

discounting factor?

$\Phi(f)$: $Z \to \mathbb{R}$

$\delta\alpha$: constant function $Z \to \mathbb{R}$

Hence $\Phi(f) + \delta\alpha \in X$

Blackwell's Lemma: Under BL1 and BL2 Φ is a d_{sup}-contraction.

Further Structure: [Remember i, ii, iii]

i. We will assume that Z is endowed with the metric d_Z, w.r.t. which it is totally bounded and complete (Z is then termed compact w.r.t. the topology generated by d_Z).

ii. It is possible to prove that if Z, d_Z is compact then $C(Z,\mathbb{R}) := \{f: Z \to \mathbb{R},\ f\ d_u/d_Z\text{-continuous}\}$ is a closed subset of $B(Z,\mathbb{R})$ w.r.t. d_{sup}. Since $B(Z,\mathbb{R}), d_{sup}$ is complete, this implies that $C(Z,\mathbb{R}), d_{sup}$ is a <u>complete</u>, metric subspace.

iii. If Z, d_Z is compact and $f \in C(Z,\mathbb{R})$ then $\underset{x\in Z}{\operatorname{argmax}} f(x) \neq \emptyset$.

Application: Bellman Equation

Z, d_Z is compact. $\omega: Z\times Z \to \mathbb{R}$ is jointly continuous, $\delta \in (0,1)$.

Definition: [Bellman Equation]

$$\forall x \in Z$$

$$(*) \qquad f(x) = \sup_{\underline{\underline{y\in Z}}} \left[\omega(x,y) + \underline{\underline{\delta}} f(y)\right], \ f \in C(Z,\mathbb{R}).$$

Lemma. Bellman equation has a unique solution inside $C(Z,\mathbb{R})$.

Remark: $\omega(x,y)+\delta f(y)$ is jointly continuous due to the continuity of ω, f. For any x, $\omega(x,y)+\delta f(y)$ is then continuous w.r.t. y. $y\in Z$, and Z is compact hence due to iii

$\underset{y\in Z}{\operatorname{argmax}}[\omega(x,y)+\delta f(y)] \neq \emptyset$ $\forall x\in X$ and thereby not only the

Supremum in the equation exists and $\sup_{y\in Z}[\omega(x,y)+\delta f(y)] =$

$\max_{y\in Z}[\omega(x,y)+\delta f(y)]$ $\forall x\in Z$. Since sup in this context is

Continuous (Remember our first application!) $\max_{y\in Z}[\omega(x,y)+\delta f(y)]$

is a continuous function of x, i.e. $\max_{y\in Z}[\omega(x,y)+\delta f(y)] \in C(Z,\mathbb{R})$.

□

Proof of Lemma.

$X = C(Z,\mathbb{R})$ and $d = d_{sup}$. (this makes sense due to ii).

Consider Φ defined on $C(Z,\mathbb{R})$ as:

$$(\Phi(f))(x) := \max_{y\in Z}[\omega(x,y)+\delta f(y)], \quad f\in C(Z,\mathbb{R}), \; x\in Z.$$

Due to the previous remark Φ is well-defined as a function $C(Z,\mathbb{R}) \to C(Z,\mathbb{R})$ (i.e. it is a self map on $C(Z,\mathbb{R})$).

Furthermore
$$(*) \iff f(x) = \max_{y\in Z}[\omega(x,y)+\delta f(y)] \quad \forall x\in Z$$
$$\iff f(x) = (\Phi(f))(x) \quad \forall x\in Z$$

$$\Leftrightarrow f = \Phi(f)$$

Hence f satisfies Bellman Equation iff it is a fixed point of Φ.

Hence in order to prove the Lemma it suffices to prove that Φ has a unique fixed point. We will use the BFPT.

α. We have that $(C(Z,\mathbb{R}), d_{sup})$ is complete due to ii above.

b. we will use Blackwell's Lemma in order to show that Φ is a d_{sup} - contraction:

I. constant functions $Z \to \mathbb{R}$ are obviously continuous and and if f continuous and α constant then $f+\alpha$ continuous $Z \to \mathbb{R}$.

II. BL1: Let $f, g \in C(Z,\mathbb{R})$ and $f \geq g \Leftrightarrow$

$$f(y) \geq g(y) \ \forall y \in Z \underset{\delta > 0}{\Leftrightarrow} \delta f(y) \geq \delta g(y) \ \forall y \in Z$$

$$\Rightarrow w(x,y) + \delta f(y) \geq w(x,y) + \delta g(y) \quad \forall x, y \in Z$$

$$\Rightarrow \sup_{y \in Z} [w(x,y) + \delta f(y)] \geq \sup_{y \in Z} [w(x,y) + \delta g(y)] \ \forall x \in Z$$

Monotonicity of sup

$$\Rightarrow (\Phi(f))(x) \geq (\Phi(g))(x) \ \forall x \in Z \Leftrightarrow \Phi(f) \geq \Phi(g).$$

Since f, g are arbitrary, Φ is Monotone and BL1 holds.

III. BL2. For the δ that appears in the equation, if $f \in C(Z, \mathbb{R})$, $\alpha \in \mathbb{R}$ we have

$$\big(\Phi(f+\alpha)\big)(x) = \sup_{y \in Z} \left[w(x,y) + \delta(f(y)+\alpha) \right] \quad \forall x \in Z$$

$$= \sup_{y \in Z} \left[w(x,y) + \delta f(y) + \underline{\delta \alpha} \right] \quad \forall x \in Z$$

$$= \sup_{y \in Z} \left[w(x,y) + \delta f(y) \right] + \delta \alpha \quad \forall x \in Z$$

$\delta \alpha$ is independent of y

$$= \big(\Phi(f)\big)(x) + \delta \alpha \quad \forall x \in Z \quad [\text{holds as equality}]$$

Hence we have shown that: $\big(\Phi(f+\alpha)\big)(x) = \big(\Phi(f)\big)(x) + \delta\alpha \;\; \forall x \in Z$

$$\Leftrightarrow \Phi(f+\alpha) = \Phi(f) + \delta\alpha$$

Hence BL2 holds as equality.

Hence Blackwell's lemma applies and thereby Φ is a $d_{\sup}$ contraction.

Hence BFPT applies and thereby Φ has a unique fixed point, and thereby Bellman equation has a unique solution. $\square$

Application: Picard - Lindelöf Theorem

- $Z = [\alpha, \beta]$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$. Z is compact w.r.t. d_a.

- $H : [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $\exists\ \delta > 0$: $\forall x \in [\alpha, \beta]$, $\forall y_1, y_2 \in \mathbb{R}$ we have:

$$|H(x, y_1) - H(x, y_2)| \leq \delta |y_1 - y_2|$$

($H(x, y)$ is Lipschitz continuous w.r.t. y, $\forall x \in X$ and the Lipschitz coefficient δ, does not depend on x.)

- $x_0 \in [\alpha, \beta]$, $y_0 \in \mathbb{R}$.

Consider the following Boundary Value Problem (BVP):

$$(*) \quad \begin{cases} f'(x) = H(x, \underline{\underline{f(x)}}), & \forall x \in [\alpha, \beta] \\ f(x_0) = y_0 \end{cases}$$

Picard - Lindelöf Theorem. There exists a unique $f \in C([\alpha, \beta], \mathbb{R})$ that solves $(*)$.

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Applications of BFPT: Picard-Lindelöf Theorem

Reminder: - $Z = [\alpha, \beta]$ $d_Z = d_u$, compactness

- $H: Z \times \mathbb{R} \to \mathbb{R}$, jointly continuous, $\exists \delta > 0$:

$\forall x \in Z, y_1, y_2 \in \mathbb{R}$

$$|H(\overset{z}{x}, \overset{f(z)}{y_1}) - H(\overset{z}{x}, \overset{g(z)}{y_2})| \leq \delta |y_1 - y_2| \quad \checkmark$$

[Lipschitz continuity of $H(x,y)$ w.r.t. y uniformly in x.]

- $x_0 \in Z$, $y_0 \in \mathbb{R}$

$*$ BVP:
$$\begin{cases} f'(x) = H(x, f(x)), \ \forall x \in [\alpha, \beta] & \checkmark \\ f(x_0) = y_0 & \checkmark \end{cases}$$

P-L Theorem: BVP has a unique solution in $C([\alpha, \beta], \mathbb{R})$

Proof: (We will use our BFPT Methodology. At first it would be helpfull if we obtained an equivalent formulation of BVP that includes only f instead of f' - this way we could hope to identify some Φ the fixedpoints of which match the solutions of BVP.)

We thus integrate $(*)$. Then:

$$(*) \iff \begin{cases} \int_{x_0}^{x} f'(z)\,dz = \int_{x_0}^{x} H(z,f(z))\,dz & \forall x\in[\alpha,\beta] \\ f(x_0) = y_0 \end{cases}$$

$$\iff \begin{cases} f(x) - f(x_0) = \int_{x_0}^{x} H(z,f(z))\,dz & \forall x\in[\alpha,\beta] \\ f(x_0) = y_0 \end{cases}$$

$$\iff \begin{cases} f(x) = f(x_0) + \int_{x_0}^{x} H(z,f(z))\,dz & \forall x\in[\alpha,\beta] \\ f(x_0) = y_0 \end{cases}$$

$$\iff f(x) = y_0 + \int_{x_0}^{x} H(z,f(z))\,dz \quad \forall x\in[\alpha,\beta] \quad (**)$$

Remark: the integral equation $(**)$ is equivalent to the BVP (BVP and $(**)$ have the same solutions). It suffices to prove that $(**)$ has a unique solution in $C([\alpha,\beta],\mathbb{R})$. $(**)$ contains f and not f' in both sides.

Remark: Consider $y_0 + \int_{x_0}^{x} H(z,f(z))\,dz$. Due to the continuity of H and f, $\int_{x_0}^{x} H(z,f(z)) \in \mathbb{R}$ $\forall x\in[\alpha,\beta]$, $\forall f\in C([\alpha,\beta],\mathbb{R})$. Hence $y_0 + \int_{x_0}^{x} H(z,f(z))\,dz$ is a well-defined function $[\alpha,\beta]\to\mathbb{R}$, $\forall f\in C([\alpha,\beta],\mathbb{R})$. $\forall x_n, x\in[\alpha,\beta]$, $x_n\to x$, $\int_{x_0}^{x_n} H(z,f(z))\,dz \to \int_{x_0}^{x} H(z,f(z))\,dz$.

Hence $y_0 + \int_{x_0}^{x} H(z,f(z))\,dz \in C([\alpha,\beta],\mathbb{R})$.

Define Φ on $C([\alpha,\beta],\mathbb{R})$ as follows:

if $f \in C([\alpha,\beta],\mathbb{R})$, $\big(\Phi(f)\big)(x) := y_0 + \int_{x_0}^{x} H(z,f(z))dz$, $\forall x \in [\alpha,\beta]$.

Due to the previous remark $\Phi(f)$ is a well-defined function $[\alpha,\beta] \to \mathbb{R}$ that is also continuous. Hence Φ is a self map on $C([\alpha,\beta],\mathbb{R})$ (i.e. $\Phi: C([\alpha,\beta],\mathbb{R}) \to C([\alpha,\beta],\mathbb{R})$).

Furthermore f is a fixed point of Φ $\Leftrightarrow$

$$f = \Phi(f) \Leftrightarrow$$

$$f(x) = \big(\Phi(f)\big)(x) \quad \forall x \in [\alpha,\beta] \Leftrightarrow$$

$$f(x) = y_0 + \int_{x_0}^{x} H(z,f(z))dz, \quad \forall x \in [\alpha,\beta] \Leftrightarrow$$

f is a solution of $(**)$.

Hence in order to prove P-L Theorem it suffices to prove that Φ has a unique fixed point. We will use BFPT:

We have that:

- $X = C([\alpha,\beta],\mathbb{R}) \subseteq B([\alpha,\beta],\mathbb{R})$
  $[\alpha,\beta]$ is compact
- We know from previous remarks (ii) that $C([\alpha,\beta],\mathbb{R}), d_{sup}$ is complete. It is possible to show (exercise!!!) that Φ is a d_{sup}-contraction under further restrictions on $x_0, \delta, \alpha, \beta$ which

do not appear in the P-L Theorem. Such restrictions would constitute a "local" version of the theorem.

We will work with d^*_{sup} defined as: $f,g \in C([a,b],\mathbb{R})$

depends on BVP

$$d^*_{sup}(f,g) = \sup_{x\in[a,b]}\left(e^{-\delta(x-x_0)}\,|f(x)-g(x)|\right)$$

$>0 \ \forall x \in [a,b]$

d^*_{sup} is a "modification" of d_{sup} in a way that "reflects" properties of the BVP.

Exercise: Show that d^*_{sup} is a well-defined metric.

We thus endow $C([a,b],\mathbb{R})$ with d^*_{sup}.

1. **Completeness:**

$$e^{-\delta(b-x_0)}\,|f(x)-g(x)| \le e^{-\delta(x-x_0)}\,|f(x)-g(x)|$$

$$\le e^{-\delta(a-x_0)}\,|f(x)-g(x)| \quad \forall x\in[a,b]$$

$$\Rightarrow \sup_{x\in[a,b]} \overbrace{e^{-\delta(b-x_0)}}^{>0}|f(x)-g(x)| \le \sup_{x\in[a,b]} e^{-\delta(x-x_0)}|f(x)-g(x)|$$

$$\le \sup_{x\in[a,b]} \overbrace{e^{-\delta(a-x_0)}}^{>0}|f(x)-g(x)| \qquad \Leftrightarrow$$

$$e^{-\delta(b-x_0)}\underbrace{\sup_{x\in[a,b]}|f(x)-g(x)|}_{d_{sup}(f,g)} \le d^*_{sup}(f,g) \le e^{-\delta(a-x_0)}\sup_{x\in[a,b]}|f(x)-g(x)|$$

$$\Leftrightarrow e^{-\delta(b-x_0)}\,d_{sup}(f,g) \le d^*_{sup}(f,g) \le e^{-\delta(a-x_0)}\,d_{sup}(f,g)$$

Hence we have: $C_1 d_{sup} \le d^*_{sup} \le C_2 d_{sup}$, $\quad C_1 = e^{-\delta(b-x_0)}$, $C_2 = e^{-\delta(a-x_0)}$

Then we know $C([\alpha,\beta],\mathbb{R}), d_{sup}$ is complete $\Longrightarrow$

due to our results on completeness

$C([\alpha,\beta],\mathbb{R}), d^*_{sup}$ is complete due to the previous inequalities.

(This essentially justifies all our previous work on Metric comparisons).

2. Contractivity: we will use the definition, we cannot use Blackwell's Lemma since we are working with d^*_{sup}.

Let $f,g \in C([\alpha,\beta],\mathbb{R})$, we have that

$$d^*_{sup}(\Phi(f),\Phi(g)) = \sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\left|(\Phi(f))(x)-(\Phi(g))(x)\right|\right]$$

$$=\sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\left|y_0+\int_{x_0}^{x}H(z,f(z))dz-\left(y_0+\int_{x_0}^{x}H(z,g(z))\,dz\right)\right|\right]$$

$$=\sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\left|\int_{x_0}^{x}(H(z,f(z))-H(z,g(z)))\,dz\right|\right]$$

$$\leq \sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\int_{x_0}^{x}\left|H(z,f(z))-H(z,g(z))\right|dz\right]$$

Lipschitz property + integral Monotonicity

$$\leq \sup_{x\in[\alpha,\beta]}\left[e^{-\delta(x-x_0)}\int_{x_0}^{x}\delta\left|f(z)-g(z)\right|dz\right]$$

$$= \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} \delta\, e^{-\delta(z-x_0)} e^{\delta(z-x_0)} |f(z)-g(z)|\, dz \right]$$

take sup w.r.t. z

$$= \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} e^{-\delta(z-x_0)} |f(z)-g(z)|\, \delta e^{\delta(z-x_0)}\, dz \right]$$

Monotonicity of integral

$$\leq \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} \underbrace{\left(\sup_{z\in[\alpha,\beta]} \left(e^{-\delta(z-x_0)} |f(z)-g(z)| \right) \right)}_{d^*_{\sup}(f,g)} \delta e^{\delta(z-x_0)}\, dz \right]$$

$$= \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} d^*_{\sup}(f,g)\, \delta e^{\delta(z-x_0)}\, dz \right]$$

$d^*_{\sup}(f,g) \geq 0$ and independent of z, x

$$= d^*_{\sup}(f,g) \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \int_{x_0}^{x} \delta e^{\delta(z-x_0)}\, dz \right]$$

$\left(e^{\delta(z-x_0)}\right)' = \delta e^{\delta(z-x_0)}$

$$= d^*_{\sup}(f,g) \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)}\, e^{\delta(z-x_0)} \Big|_{x_0}^{x} \right] =$$

$$= d^*_{\sup}(f,g) \sup_{x\in[\alpha,\beta]} \left[e^{-\delta(x-x_0)} \left(e^{\delta(x-x_0)} - 1 \right) \right]$$

$$= d^*_{\sup}(f,g) \sup_{x\in[\alpha,\beta]} \left(1 - e^{-\delta(x-x_0)} \right) = \left(1 - e^{-\delta(\beta-x_0)} \right) d^*_{\sup}(f,g)$$

We have shown that: $\forall f,g \in C([a,b],\mathbb{R})$

$$d^*_{\sup}(\Phi(f),\Phi(g)) \leq \left(1-e^{-\delta(b-x_0)}\right) d^*_{\sup}(f,g)$$

We have that $0<e^{-\delta(b-x_0)}\leq 1 \iff 0\leq 1-e^{-\delta(b-x_0)}<1$

Hence the $d^*_{\sup}$ - Lipschitz coefficient of Φ is <1

and Φ is a $d^*_{\sup}$ - contraction.

Hence due to the BFPT Φ has a unique fixed point $\iff$ (**) has a unique solution $\iff$ BVP has a unique solution. $\square$