# Lecture 12

BFPT and Applications:

- Bellman Equation

**Banach Fixed Point Theorem** (BFPT). Suppose that $(X,d)$ is complete, and $f: X \to X$, is a $d$-contraction then $f$ has a unique fixed point (say $x^*$), and $\forall x \in X$ for $x_n = \begin{cases} x, & n=0 \\ f(x_{n-1}), & n>0 \end{cases} = \begin{cases} x, & n=0 \\ f^{(n)}(x), & n>0 \end{cases} = f^{(n)}(x)$,

$\lim_{n\to+\infty} x_n = x^*$ (w.r.t. $d$).

Remark: BFPT is very powerfull: it asks a lot but it also gives uniqueness and approximation of the fixed point.

Proof. The proof will be based on the following lemmata:

**Lemma (Uniqueness).** If $f$ is $d$-contraction and $f$ has a fixed point then this is unique. (this does not use completeness).

Proof. Suppose that $x^*, x^{**}$ with $x^* \neq x^{**}$ are fixed points of $f$. Then

$$0 < d(x^*, x^{**}) = d(f(x^*), f(x^{**}))$$

$\leq c_f\, d(x^*, x^{**})$ where the last inequality follows due to the fact that $f$ is a $d$-contraction.

Summarizing we have

$$0 < \underbrace{d(x^*, x^{**}) \le c_f\, d(x^*, x^{**})}$$

$$\Downarrow$$

$$d(x^*, x^{**}) - c_f\, d(x^*, x^{**}) \le 0 \iff$$

$(1-c_f)\, d(x^*, x^{**}) \le 0$. This is impossible since $1-c_f > 0$ and $d(x^*, x^{**}) > 0$. Hence $f$ has a unique fixed point. $\square$

**Lemma (Fixed Points as Limits)** If $(x_n)$ (see above for the definition) has a limit w.r.t. $d$, say $y$, then $y$ is a fixed point of $f$. (completeness in not used here)

**Proof.** We have that $y = \lim_{n\to+\infty} x_n = \lim_{n\to+\infty} f^{(n)}(x)$

$$\overset{\text{w.r.t. } d}{=} \lim_{n\to+\infty} f(f^{(n-1)}(x)) = f(\lim_{n\to+\infty} f(x_{n-1})) = f(\lim_{n\to+\infty} x_{n-1})$$

$f$ is contraction $\Downarrow$ Lipschitz cont. $\Downarrow$ cont.

$$\hookrightarrow = f(\lim_{n\to+\infty} f^{(n-1)}(x))$$

$$\overset{x_n \to y}{=} f(y) \Rightarrow y = f(y)$$

$x_{n-1} \to y$ (why?)

hence $y = \lim_{n \to +\infty} x_n$ is a fixed point of $f$. $\square$

**Lemma (Technical).** $\forall n \geq 0 \quad d(x_{n+1}, x_n) \leq C_f^n d(x_1, x_0)$.

*Proof.* By induction:

a. It is true for $n=0$ since

$$d(x_1, x_0) = d(x_1, x_0) = C_f^0 d(x_1, x_0).$$

b. Suppose that it is true for $n=k$, i.e.

$$d(x_{k+1}, x_k) \leq C_f^k d(x_1, x_0) \quad (*).$$

c. For $n=k+1$ we have

$$d(x_{k+2}, x_{k+1}) = d(f(x_{k+1}), f(x_k)) \underset{f \text{ contraction}}{\leq}$$

$$C_f \, d(x_{k+1}, x_k) \underset{(*)}{\leq} C_f \cdot C_f^k d(x_1, x_0)$$

$$= C_f^{k+1} d(x_1, x_0). \Rightarrow$$

$d(x_{k+2}, x_{k+1}) \leq C_f^{k+1} d(x_1, x_0)$ hence the Property is true for $n=k+1$. $\square$

# Lecture 12

**Lemma (Cauchyness).** $(x_n)$ is $(d\text{-})$ Cauchy.

*Proof.* We have to examine the limiting behavior of $d(x_m, x_n)$ as $\min(m,n) \to +\infty$.

Suppose first that $m>n$ : due to triangle inequality

$$d(x_m,x_n) \leq d(x_m,x_{m-1}) + d(x_{m-1},x_n)$$

$$\leq d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + d(x_{m-2},x_n)$$

$$\dots \leq d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \dots + d(x_{n+1},x_n)$$

$$\leq c_f^{m-1} d(x_1,x_0) + c_f^{m-2} d(x_1,x_0) + \dots + c_f^{n} d(x_1,x_0)$$

Technical

$$= c_f^n d(x_1,x_0) \left( c_f^{m-1-n} + c_f^{m-2-n} + \dots + 1 \right) =$$

$$= c_f^n d(x_1,x_0) \sum_{i=0}^{m-1-n} c_f^i \underset{c_f \geq 0}{\leq} c_f^n d(x_1,x_0) \sum_{i=0}^{\infty} c_f^i$$

$$\underset{0 \leq c_f < 1}{=} \frac{c_f^n}{1-c_f} d(x_1,x_0)$$

$$\sum_{i=0}^{\infty} c_f^i = \frac{1}{1-c_f}$$

geometric series

We have shown that if $m>n$, $d(x_m,x_n) \leq \frac{c_f^n}{1-c_f} d(x_1,x_0)$

It is obvious (by interchanging $m$ with $n$) that

when $n \geq m$ $d(x_m,x_n) \leq \frac{c_f^m}{1-c_f} d(x_1,x_0)$.

Hence we generally obtain that:

$$(**) \quad d(x_m,x_n) \leq \frac{c_f^{\min(m,n)}}{1-c_f} d(x_1,x_0), \ \forall m,n \in \mathbb{N}$$

We have that due to $(**)$ and that $d(x_m,x_n) \geq 0$ $\forall m,n \in \mathbb{N}$

$$0 \le \lim_{\min(m,n)\to+\infty} d(x_m,x_n) \le \lim_{\min(m,n)\to+\infty} \frac{C_f^{\min(m,n)}}{1-C_f} d(x_1,x_0)$$

$$= \frac{d(x_1,x_0)}{1-C_f} \lim_{\min(m,n)\to+\infty} C_f^{\min(m,n)} \underset{0\le C_f<1}{=} \frac{d(x_1,x_0)}{1-C_f} \cdot 0 = 0.$$

Hence $(x_m)$ is $d$-Cauchy. $\square$

**Proof of BFPT continued:** $\forall x \in X$, $x_m = f^{(m)}(x)$, $(x_m)$ is a Cauchy sequence due Lemma Cauchyness. Since $(X,d)$ is complete, $(x_m)$ is $(d-)$ convergent by being Cauchy. Due to Lemma Fixed points as limits, the $(d-)$ limit of $(x_m)$ is a fixed point of $f$. Due to Lemma Uniqueness $f$ has a unique fixed point. Finally due to Lemma fixed points as limits, this unique fixed point is the limit of $(x_m)$. $\square$

**Corollary.** $(X,d)$ is complete and for some $m>0$, $f^{(m)}$ is $(d-)$ contraction. Then $f$ has a unique fixed point.

*Proof.* By BFPT $f^{(m)}$ has a unique fixedpoint. Then due to Remark $f$ has a unique fixed point. $\square$

*Example.* Suppose that we want to study the equation (in $\mathbb{R}$)

$x = \sin(x)$. $x$ is a solution iff $x$ is a fixed point of the sin function. Hence we study the fixed point properties of this function. We know that $\mathbb{R}, d_u$ is complete. Is sin a $d_u$-contraction? Due to the results in the previous lecture sin is a $d_u$ contraction iff $C_{\sin} := \sup_{x \in \mathbb{R}} \left| \frac{d\sin(x)}{dx} \right| < 1$

But $\sup_{x \in \mathbb{R}} |\cos(x)| = 1$ hence sin is $d_u/d_u$-Lipschitz continuous, but not a $d_u$-contraction. Consider $\sin^{(2)}(x) := \sin(\sin(x))$. Show that $C_{\sin^{(2)}} = \sup_{x \in \mathbb{R}} \left| \frac{d\sin(\sin(x))}{dx} \right| < 1$

Hence $\sin^{(2)}$ is contractive. Hence BFPT is applicable and thereb $\sin^{(2)}$ has a unique fixed point. Hence $\sin(x)$ has a unique fixed point, hence $x = \sin(x)$ has a unique solution. $\square$

[redacted]

Reminder:

* BFPT: If $X$ admits a $d$, such that i. $(X,d)$ is complete, and ii. $f: X \to X$ is a $d$-contraction, then $f$ has a unique fixed point in $X$, say $x^*$, and $\forall y \in X$, $x^* = \lim_{n \to \infty} x_n$ (w.r.t. $d$), where $x_n = f^{(n)}(y)$.

Corollary [Further locating $x^*$]. If $A$ is a closed and non empty subset of $X$, and $\underline{\forall x \in A,\ f(x) \in A}$, then $x^* \in A$.

↳ $f$ remains a self map when restricted to $A$.

Proof. $X,d$ is complete and $A$ is closed (w.r.t. $d$) subset of $X$. Hence $(A,d)$ is a complete metric subspace of $X$. When $f$ is restricted to $A$ it remains a self-map. $f: X \to X$ is a $d$-contraction, hence it is also a $d$-contraction when restricted to $A$. Hence due to BFPT $f$ restricted to $A$ has a unique fixed point inside $A$. But we also know that $f$ has a unique fixed point, $x^*$, in $X \supseteq A$. Hence $x^* \in A$. □

Corollary. If $A$ is closed and $x^* \notin A$ then $\exists x \in A$: $f(x) \notin A$.

# Applications of BFPT:

General Framework:

- $X$ will be generally a subset of $B(Z,\mathbb{R})$ for $Z$ some set and $d$ will be either $d_{sup}$ or some "modification" of $d_{sup}$.
- $\Phi$ will denote the respective self-map on $X$. (i.e. if $f$ lies in the aforementioned function space, $\Phi(f)$ will also lie there).
- We will study problems of existence and uniqueness of solutions of equations on such function spaces (functional equations)
- The crucial step will be to represent the issue of existence and uniqueness of solutions by the issue of the existence and uniqueness of fixed points of a suitable $\Phi$. In this respect we will have to:
  * identify $\Phi$ as the self-map the fixed points of which are the solutions of the equations
  * show that $\Phi$ is well-defined (i.e. that it is a self-map)
  * ensure that BFPT is applicable in this framework.

- One possible difficulty is to show that $\Phi$ is a contraction. One way to show contractivity by avoiding the definition goes through Blackwell's lemma. This runs as follows:

— $d = d_{sup}$

— $X \subseteq B(Z, \mathbb{R})$ contains every constant function.

Furthermore, if $f \in X$, $\alpha \in \mathbb{R}$ then $f + \alpha \in X$.

($\alpha$ can be perceived as a constant function $Z \to \mathbb{R}$, $\alpha(x) = \alpha$ $\forall x \in Z$)

$(f+\alpha)(x) = f(x) + \alpha(x)$
$= f(x) + \alpha$

(e.g. $B(Z, \mathbb{R})$ satisfies the above)

— if $f, g \in X$ then $f \gtrsim g \Leftrightarrow f(x) \geq g(x)$ $\forall x \in Z$. (Definition)

(partial order on $X$)

Blackwell Hypotheses:

BL1. if $f \gtrsim g \Rightarrow \phi(f) \gtrsim \phi(g)$ (Monotonicity of $\phi$)

$\Updownarrow$

$\left[ \begin{array}{l} \phi(f), \phi(g) : Z \to \mathbb{R} \in X \\ (\phi(f))(x) \geq (\phi(g))(x), \ \forall x \in Z \end{array} \right]$

BL2. $\exists\, \delta \in (0,1) : \forall f \in X, \alpha \in \mathbb{R}$

$$\phi(f) + \delta\alpha \gtrsim \phi(f + \alpha)$$

discounting factor?

$Z \to \mathbb{R}$ constant function $Z \to \mathbb{R}$

Hence $\phi(f) + \delta\alpha \in X$

Blackwell's Lemma: Under BL1 and BL2 $\phi$ is a $d_{sup}$-contraction.

Proof. Let $f, g \in X$. $f(x) - g(x) \leq |f(x) - g(x)| \leq$
$\sup_{x \in Z} |f(x) - g(x)| = d_{sup}(f, g)$ $\forall x \in Z$

$\Longrightarrow \quad f(x) \leq g(x) + d_{sup}(f,g) \quad \forall x \in Z \quad (*)$

Define $g^*: Z \to \mathbb{R}$, by $g^*(x) := g(x) + d_{sup}(f,g)$, $\forall x \in Z$

then $g^* \in X$.

Due to $(*)$ $\quad g^* \gtrsim f$.

Analogously we can show that:

$$g(x) \leq f(x) + d_{sup}(f,g) \quad \forall x \in Z$$

Define $f^*: Z \to \mathbb{R}$, by $f^*(x) := f(x) + d_{sup}(f,g)$, $\forall x \in Z$

and $f^* \in X$.

Analogously $\quad f^* \gtrsim g$.

Hence by the above constructions we have:

$$\left\{\begin{array}{l} g^* \gtrsim f \\ f^* \gtrsim g \end{array}\right. \xRightarrow[\substack{\phi \text{ is monotone} \\ \text{by BL1}}]{f,g,f^*,g^* \in X} \left\{\begin{array}{l} \phi(g^*) \gtrsim \phi(f) \\ \phi(f^*) \gtrsim \phi(g) \end{array}\right. \quad (*) \quad \in X$$

$$\phi(g^*) = \phi(g + d_{sup}(f,g)) \underset{BL2}{\lesssim} \phi(g) + \delta d_{sup}(f,g)$$

$$\phi(f^*) = \phi(f + d_{sup}(f,g)) \lesssim \phi(f) + \delta d_{sup}(f,g) \quad (**)$$

$$\left.\begin{array}{l} \phi(g) + \delta d_{sup}(f,g) \gtrsim \phi(g^*) \gtrsim \phi(f) \\ \phi(f) + \delta d_{sup}(f,g) \gtrsim \phi(f^*) \gtrsim \phi(g) \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l}\Phi(g) + \delta\, d_{sup}(f,g) \gtrsim \Phi(f) \\ \Phi(f) + \delta\, d_{sup}(f,g) \gtrsim \Phi(g)\end{array}\right\} \iff$$

$$\left.\begin{array}{l}\big(\Phi(g)\big)(x) + \delta\, d_{sup}(f,g) \geq \big(\Phi(f)\big)(x) \\ \big(\Phi(f)\big)(x) + \delta\, d_{sup}(f,g) \geq \big(\Phi(g)\big)(x)\end{array}\right\} \forall x \in Z \iff$$

$$\left.\begin{array}{l}\big(\Phi(f)\big)(x) - \big(\Phi(g)\big)(x) \leq \delta\, d_{sup}(f,g) \\ \big(\Phi(g)\big)(x) - \big(\Phi(f)\big)(x) \leq \delta\, d_{sup}(f,g)\end{array}\right\} \forall x \in Z \implies$$

$$\left|\big(\Phi(f)\big)(x) - \big(\Phi(g)\big)(x)\right| \leq \delta\, d_{sup}(f,g) \quad \forall x \in Z$$

and since $\delta\, d_{sup}(f,g)$ is independent of $x$, we have that

$$\sup_{x \in Z} \left|\big(\Phi(f)\big)(x) - \big(\Phi(g)\big)(x)\right| \leq \delta\, d_{sup}(f,g) \iff$$

$$d_{sup}\big(\Phi(f), \Phi(g)\big) \leq \delta\, d_{sup}(f,g)$$

Since $\delta \in (0,1)$ and $f,g$ are arbitrary the previous shows that $\Phi$ is a $d_{sup}$-contraction as requested. $\square$

## Further Structure:

i. We will assume that $Z$ is endowed with the metric $d_Z$, w.r.t. which it is totally bounded and complete ($Z$ is then termed compact w.r.t. the topology generated by $d_Z$).

ii. It is possible to prove that if $Z, d_Z$ is compact then $C(Z,\mathbb{R}) := \{ f : Z \to \mathbb{R},\ f\ d_{\mathbb{R}}/d_Z\text{-continuous} \}$ is a closed subset of $B(Z,\mathbb{R})$ w.r.t. $d_{sup}$. Since $B(Z,\mathbb{R}), d_{sup}$ is complete, this implies that $C(Z,\mathbb{R}), d_{sup}$ is a complete, metric subspace.

iii. If $Z, d_Z$ is compact and $f \in C(Z,\mathbb{R})$ then $\operatorname{argmax}_{x \in Z} f(x) \neq \emptyset$.

## Application: Bellman Equation

End of Lecture 12

$Z, d_Z$ is compact. $\omega : Z \times Z \to \mathbb{R}$ is jointly continuous, $\delta \in (0,1)$.

**Definition:** [Bellman Equation]

$$(*) \qquad f(x) = \sup_{y \in Z} \left[ \omega(x,y) + \delta f(y) \right], \ f \in C(Z,\mathbb{R}).$$

**Lemma.** Bellman equation has a unique solution inside $C(Z,\mathbb{R})$.

Remark: $\omega(x,y)+\delta f(y)$ is jointly continuous due to the continuity of $\omega, f$. For any $x$, $\omega(x,y)+\delta f(y)$ is then continuous w.r.t. $y$. $y \in Z$, and $Z$ is compact hence due to iii

$\underset{y\in Z}{\operatorname{argmax}}[\omega(x,y)+\delta f(y)] \neq \emptyset \ \forall x \in X$ and thereby not only the

Supremum in the equation exists and $\sup_{y\in Z}[\omega(x,y)+\delta f(y)] =$
$\max_{y\in Z}[\omega(x,y)+\delta f(y)] \ \forall x \in Z$. Since sup in this context is

continuous (Remember our first application!) $\max_{y\in Z}[\omega(x,y)+\delta f(y)]$
is a continuous function of $x$, i.e. $\max_{y\in Z}[\omega(x,y)+\delta f(y)] \in C(Z,\mathbb{R})$. $\square$

Proof of Lemma.

$X = C(Z,\mathbb{R})$ and $d = d_{sup}$. (this makes sense due to ii).

Consider $\Phi$ defined on $C(Z,\mathbb{R})$ as:

$$(\Phi(f))(x) := \max_{y\in Z}[\omega(x,y)+\delta f(y)], \quad f \in C(Z,\mathbb{R}), \ x \in Z.$$

Due to the previous remark $\Phi$ is well-defined as a function $C(Z,\mathbb{R}) \to C(Z,\mathbb{R})$ (i.e. it is a self map on $C(Z,\mathbb{R})$).

Furthermore $(*) \iff f(x) = \max_{y\in Z}[\omega(x,y)+\delta f(y)] \ \forall x \in Z$

$\iff f(x) = (\Phi(f))(x) \ \forall x \in Z$

$$\Leftrightarrow f = \Phi(f)$$

Hence $f$ satisfies Bellman Equation iff it is a fixed point of $\Phi$.

Hence in order to prove the Lemma it suffices to prove that $\Phi$ has a unique fixed point. We will use the BFPT.

α. We have that $C(Z,\mathbb{R}), d_{sup}$ is complete due to ii above.

b. we will use Blackwell's Lemma in order to show that $\Phi$ is a $d_{sup}$ - contraction:

I. constant functions $Z \to \mathbb{R}$ are obviously continuous and and if $f$ continuous and $\alpha$ constant then $f+\alpha$ continuous $Z \to \mathbb{R}$.

II. BLL: Let $f, g \in C(Z,\mathbb{R})$ and $f \gtrsim g \Leftrightarrow$

$$f(y) \geq g(y) \ \forall y \in Z \underset{\delta > 0}{\Leftrightarrow} \delta f(y) \geq \delta g(y) \ \forall y \in Z$$

$$\Rightarrow \omega(x,y) + \delta f(y) \geq \omega(x,y) + \delta g(y) \quad \forall x,y \in Z$$

$$\Rightarrow \sup_{y \in Z}[\omega(x,y) + \delta f(y)] \geq \sup_{y \in Z}[\omega(x,y) + \delta g(y)] \ \forall x \in Z$$

Monotonicity of sup

$$\Rightarrow (\Phi(f))(x) \geq (\Phi(g))(x) \ \forall x \in Z \Leftrightarrow \Phi(f) \gtrsim \Phi(g).$$

Since $f, g$ are arbitrary, $\Phi$ is Monotone and BLL holds.

III. BL2. For the $\delta$ that appears in the equation, if $f \in C(Z, \mathbb{R})$, $\alpha \in \mathbb{R}$ we have

$$(\Phi(f+\alpha))(x) = \sup_{y \in Z} [w(x,y) + \delta(f(y)+\alpha)] \quad \forall x \in Z$$

$$= \sup_{y \in Z} [w(x,y) + \delta f(y) + \underline{\underline{\delta\alpha}}] \quad \forall x \in Z$$

$$= \sup_{y \in Z} [w(x,y) + \delta f(y)] + \delta\alpha \quad \forall x \in Z$$

$\delta\alpha$ is independent of $y$

$$= (\Phi(f))(x) + \delta\alpha \quad \forall x \in Z$$

Hence we have shown that: $(\Phi(f+\alpha))(x) = (\Phi(f))(x) + \delta\alpha \;\; \forall x \in Z$

$$\Leftrightarrow \Phi(f+\alpha) = \Phi(f) + \delta\alpha$$

Hence BL2 holds as equality.

Hence Blackwell's lemma applies and thereby $\Phi$ is a $d_{sup}$ contraction.

Hence BFPT applies and thereby $\Phi$ has a unique fixed point, and thereby Bellman equation has a unique solution. $\square$

# Application: Picard - Lindelöf Theorem

- $Z = [\alpha, \beta]$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$. $Z$ is compact w.r.t. $d_a$.

- $H : [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $\exists\, \delta > 0$:
$\forall x \in [\alpha, \beta]$, $\forall y_1, y_2 \in \mathbb{R}$ we have:

$$|H(x, y_1) - H(x, y_2)| \leq \delta\, |y_1 - y_2|$$

($H(x,y)$ is Lipschitz continuous w.r.t. $y$, $\forall x \in X$ and the Lipschitz coefficient $\delta$, does not depend on $x$.)

- $x_0 \in [\alpha, \beta]$, $y_0 \in \mathbb{R}$.

Consider the following Boundary Value Problem (BVP):

$$(*) \quad \begin{cases} f'(x) = H(x, \underline{f(x)}), & \forall x \in [\alpha, \beta] \\ f(x_0) = y_0 \end{cases}$$

Picard - Lindelöf Theorem. There exists a unique $f \in C([\alpha, \beta], \mathbb{R})$ that solves $(*)$.

- $X = C([\alpha, \beta], \mathbb{R})$

- to be continued...