Lecture 10

* Cauchy Sequences - Completeness
* Lipschitz Continuity

* We need two more notions in order to obtain the nessecary background for the FPT. Those are non topological - yet closely entangled with the previous!

Further Non topological notions in Metric Spaces

1. Completeness

A. Cauchy Sequences (Βασικές Ακολουθίες)

Definition. (X,d) a m.s. and (x_n) is such that $x_n \in X \ \forall n \in \mathbb{N}$.

Then (x_n) is Cauchy iff $\forall \varepsilon>0$, $\exists n^*(\varepsilon)$: $\forall n,m \geq n^*(\varepsilon)$ $d(x_n,x_m)<\varepsilon$. [Definition Denial: $\exists \varepsilon>0$: $\forall n^* \in \mathbb{N}$, $\exists n,m \geq n^*$: $d(x_n,x_m) \geq \varepsilon$]

Remark: The above describes a property of "asymptotic concentration" for (x_n). $\forall \varepsilon>0$ $\exists n^*(\varepsilon)$:

$$\Big(x_0, x_1, x_2, \dots, \overbrace{\underbrace{x_{n^*(\varepsilon)}, x_{n^*(\varepsilon)+1}}_{\text{distance}<\varepsilon}, \dots, x_n}^{\text{distance}<\varepsilon}, x_m, \dots\Big)$$

distance $<\varepsilon$ (between $x_{n^*(\varepsilon)+1}$ and x_m)

etc.

Example. $X=\mathbb{R}$, $d=d_u$ $x_n=\frac{1}{n+1}$. Is (x_n) Cauchy?

We examine $(*)\ |x_n-x_m|<\varepsilon \iff \left|\frac{1}{n+1}-\frac{1}{m+1}\right|<\varepsilon$. Let $m^*=\max(n,m)$

$n^*=\min(n,m) \Rightarrow m^*-n^*=k\in\mathbb{N} \iff m^*=n^*+k$.

$$\left|\frac{1}{n+1}-\frac{1}{m+1}\right| = \frac{1}{n^*+1}-\frac{1}{m^*+1} = \frac{m^*+1-(n^*+1)}{(n^*+1)(m^*+1)} = \frac{m^*-n^*}{(n^*+1)(n^*+1+k)} =$$

$$= \frac{k}{(n^*+1)^2+k(n^*+1)}.$$

$$(*) \iff \frac{k}{(n^*+1)^2+k(n^*+1)} < \varepsilon \iff \frac{k}{\varepsilon} < (n^*+1)^2+k(n^*+1) \iff$$

$k(n^*+1)+(n^*+1)^2 > \frac{k}{\varepsilon}\ (**)$ (when $k=0$ $n=m$ hence $(*)$ is in any case valid)

Hence suppose $k>0$, hence $(**) \iff n^*+1+\frac{(n^*+1)^2}{k} > \frac{1}{\varepsilon}$

$\iff n^* + \cancel{\frac{(n^*+1)^2}{k}} > \frac{1}{\varepsilon} - 1$. $(***)$. The $(***)$ is valid whenever $n^*>\frac{1}{\varepsilon}-1$ and the latter holds

$\forall n^*\in\mathbb{N}$: $n^*\geq$ smallest Natural greatest than or equal to $\frac{1}{\varepsilon}-1$

Hence if we define $n^*(\varepsilon):=$ smallest natural $\geq \frac{1}{\varepsilon}-1$ we have that $\forall n,m\geq n^*(\varepsilon)$ $|x_n-x_m|<\varepsilon$. Since ε is arbitrary this implies that $\left(\frac{1}{n+1}\right)$ is Cauchy. □

$$0 \leq \lim_{\min(n,m)\to+\infty} d_u(x_m,x_n) = \lim_{\min(n,m)\to+\infty} \left|\frac{1}{m+1}+\frac{1}{n+1}\right| \leq \lim_{m\to+\infty}\underbrace{\frac{1}{m+1}}_{=0}+\lim_{n\to+\infty}\underbrace{\frac{1}{n+1}}_{=0} = 0$$

Counter-Example. $X = \mathbb{R}$, $d = d_u$, $x_n = n$. Is (n) Cauchy?

Suppose that it is. Let $\varepsilon = 1/3$. Then $\exists n^*(1/3)$: $\forall n,m \geq n^*(1/3)$

$|x_n - x_m| < 1/3$. Set $n := n^*(1/3)$, $m := n^*(1/3) + 1$. So we must

have that $|x_n - x_m| = |n^*(1/3) - (n^*(1/3) + 1)| = 1$ and

$1 < 1/3$, impossible. Hence (n) is not Cauchy. □

Lemma. If (x_n) converges then it is Cauchy. (w.r.t. d)

Proof. Let $\varepsilon > 0$. Let x be the limit of (x_n). $\exists\, n(\varepsilon/2)$:

$\forall n \geq n(\varepsilon/2)$ $d(x, x_n) < \varepsilon/2$. $\forall n,m \geq n(\varepsilon/2)$, $d(x_n, x_m) \leq \underbrace{d(x, x_n)}_{<\varepsilon/2} + \underbrace{d(x, x_m)}_{<\varepsilon/2}$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $\forall n,m \geq n(\varepsilon/2)$, $d(x_n, x_m) < \varepsilon$. Set $n^*(\varepsilon) := n(\varepsilon/2)$.

The result follows since ε is arbitrary. □

Remark. We have shown that the set of convergent sequences $\subseteq$ the set of Cauchy sequences in (X, d).

The reverse implication does not hold as the following example shows:

Example. $X = (0,1]$, $d = d_u$ restricted to $(0,1]$. $x_n = 1/(n+1) \in (0,1]$

then. ((x_n) also belongs to the particular METRIC SUBSPACE of $(\mathbb{R}, d_u)$).

It is easy to see that $(1/(n+1))$ is Cauchy as a sequence inside $(0,1]$ — use the exact same arguments with the first example.

But $x_n \to 0 \notin X$, hence (x_n) cannot be considered as convergent in the particular X. Hence it is possible that a metric space (X,d) "does not contain the limits" of some of its Cauchy sequences. Hence it is possible that the set of convergent sequences $\subsetneq$ the set of Cauchy sequences in (X,d).

B. Complete Metric Spaces

Definition. (X,d) is complete iff every Cauchy sequence in X (w.r.t. d) is also convergent in X (w.r.t. d)

[i.e. iff the set of convergent sequences = the set of Cauchy sequences in (X,d)].

Example. It can be shown that $(\mathbb{R}, d_u)$ is complete.

Counter Example. $(\underbrace{(0,1], d_u}_{\text{restriction}})$ is not complete as the previous example shows. [hence restriction may destroy completeness]

Example. X is general $d = d_d$. (x_n) is Cauchy (w.r.t. d_d) iff it is eventually constant. (Suppose that it is eventually constant i.e. it is of the form $(x_0, x_1, \ldots, x_k, c, c, c, \ldots, c, \ldots)$ for some $k \in \mathbb{N}$ and $c \in X$. $\forall \varepsilon > 0$ $d(x_n, x_m) = 0 < \varepsilon$ $\forall n, m \geq k+1$, hence it is Cauchy) (Suppose that (x_n) is (d_d)-Cauchy then for $\varepsilon = 1/2$, $\exists n^*(1/2): \forall n,m \geq n^*(1/2), d_d(x_n, x_m) < 1/2$

$\Leftrightarrow x_n = x_m \; \forall n,m \geq n^*(1/2)$ hence (x_n) is eventually constant). We have already proven that (x_n) is $(d_d$-$)$ convergent iff it is eventually constant. Hence

Set of Cauchy Sequences = Set of eventually constant sequences = Set of convergent sequences in (X, d_d).

Hence every discrete space is complete!

(Remember Cauchyness and Completeness also depend on the metric!)

When is the completeness property inherited by metric subspaces?

Let (X,d) is a complete metric space. $A \subseteq X$, $A \neq \emptyset$, we consider the metric subspace (A,d). The subspace is complete iff A is a closed subset of X.

restriction

Proof. (if (A,d) is complete then A is closed) Let (x_n) be such that $x_n \in A \; \forall n \in \mathbb{N}$, and $x_n \to x \in X$. Since (x_n) is convergent in X it is Cauchy in X. Hence it is Cauchy in A. Since (A,d) is complete (x_n) converges in A. Hence by the uniqueness of limits $x \in A$. Since (x_n) is arbitrary A is closed.

(if A is closed then (A,d) is complete). Suppose that (x_n) is Cauchy in A. Hence (x_n) is Cauchy in X. Since (X,d) is complete then (x_n) converges in X. Since A is closed (x_n) converges in A. Since (x_n) is arbitrary every Cauchy sequence in A converges in A. Hence (A,d) is complete. $\square$

[completeness inheritance]

Example. Let $X = B(Z,Y)$, (Y, d_Y), $d^Y_{sup}(f,g) = \sup_{x\in Z} d_Y(f(x),g(x))$. It is possible to show that if Y, d_Y is complete then $(B(Z,Y), d^Y_{sup})$ is complete. We know that $(\mathbb{R}, d_u)$ is complete then by the previous shows that $(B(Z,\mathbb{R}), d_{sup})$ is complete. (see tutorial for the proof).

C. Cauchyness – Completeness and Metrics Comparison

X general, d_1, d_2 metrics and $\exists c>0 : d_1 \leq c d_2$.

- If (x_n) is Cauchy w.r.t. d_2 then it is Cauchy w.r.t. d_1 (Exercise)

- Hence set of Cauchy sequences in (X,d_2) $\subseteq$ Set of Cauchy sequences in (X,d_1) $\square$

[redacted]

Reminder:

* (x_n) is $(d-)$ Cauchy iff, $\forall \varepsilon>0\ \exists n^*(\varepsilon)$:

$\forall n,m \geq n^*(\varepsilon) \quad d(x_n,x_m)<\varepsilon$

$\iff \forall \varepsilon>0\ \exists n^*(\varepsilon)\ \forall n,m\geq n^*(\varepsilon)\ d(x_n,x_m)\in O_{d_{\mathbb{R}}}(0,\varepsilon)$

$\iff \lim\limits_{\min(n,m)\to\infty} d(x_n,x_m)=0$

* If $\exists\, c>0 : d_1\leq c d_2$ and (x_n) is d_2-Cauchy then

Since $d_1(x_n,x_m)\leq c\, d_2(x_n,x_m)\ \forall n,m \implies$

$\lim\limits_{\min(n,m)\to+\infty} d_1(x_n,x_m)\leq c \lim\limits_{\min(n,m)\to\infty} d_2(x_n,x_m)=0 \implies$

(x_n) is d_1- Cauchy.

Hence Set of d_1- Cauchy Seq $\supseteq$ Set of d_2- Cauchy Seq.

Continue:

- If $\exists\ c_1,c_2>0$: $c_1 d_2\leq d_1\leq c_2 d_2$ then (x_n) is d_1- Cauch iff (x_n) is d_2-Cauchy

[i.e. Set of d_1-Cauchy Seq = Set of d_2-Cauchy Seq]

$\Downarrow$ + result on the limits

X is d_1- Complete $\iff$ X is d_2- Complete $\square$

E.g. $X = \mathbb{R}^p$ $d = d_I$ it is possible to prove that $\mathbb{R}^p$ is d_I-complete. Hence $\mathbb{R}^p$ is d-complete for any $d = d_I, d_{max}, d_{11}$.

9. Lipschitz Continuity

* "Usual" continuity of functions between metric spaces is somewhat equivalent to the "preservation of sequential convergence".

* Is "Cauchyness" preserved by usual continuity?

Answer: No! For example: $X = (0,1]$, $d_X = d_u$ (restricted); $Y = \mathbb{R}$, $d_Y = d_u$. $f: (0,1] \to \mathbb{R}$, $f(x) := \frac{1}{x}$ (f is obviously d_u/d_u-continuous). For $x_n = \frac{1}{n+1}$, $n \in \mathbb{N}$, we know that $(1/n+1)$ is d_u-Cauchy in X. Now, $f(x_n) = \frac{1}{x_n} = \frac{1}{1/n+1} = n+1$ hence we obtain $(f(x_n)) = (n+1)$ "living" in Y. The latter is not d_u-Cauchy. Hence the "usual" continuity does not generally preserve Cauchyness.

* Is it possible to define stronger notions of continuity that "preserve" Cauchyness?

One way to do it is through the notion of Lipschitz continuity.

Framework: (X,d_X), (Y,d_Y), $f: X \to Y$.

Definition. f is (d_Y/d_X-) Lipschitz continuous iff $\exists c>0$:

$$\forall x,y \in X, \quad d_Y(f(x),f(y)) \leq c\, d_X(x,y).$$

Lemma. If f is (d_Y/d_X-) Lipschitz continuous then it is continuous.

Proof. Let $x, x_n \in X$, and $x_n \to x$ (w.r.t. d_X). It suffices to prove that $f(x_n) \to f(x)$ (w.r.t. d_Y), which is equivalent to that $\lim_{n\to+\infty} d_Y(f(x_n),f(x))=0$.

Since f is Lipschitz-continuous

$$d_Y(f(x_n),f(x)) \leq c\, d_X(x_n,x) \quad \forall n\in\mathbb{N} \Rightarrow$$

$$\underbrace{\lim_{n\to+\infty} d_Y(f(x_n),f(x))}_{\geq 0 \ \forall n\in\mathbb{N}} \leq c \lim_{n\to+\infty} d_X(x_n,x) = 0 \quad \text{since } x_n \to x$$

$$\Rightarrow \lim_{n\to+\infty} d_Y(f(x_n),f(x))=0. \quad \square$$

Remark. If c exists it is not unique, for any $c^* > c$ we have that $\forall x,y\in X$, $d_Y(f(x),f(y)) \leq c^* d(x,y)$

the infimum of positive constants for which the definitional inequality holds and it is called Lipschitz Coefficient of f (C_f). $\square$

Does Lipschitz Continuity preserve Cauchyness?

Lemma. If (x_n) is $(d_X$-$)$ Cauchy and f is Lipschitz continuous, then $(f(x_n))$ is $(d_Y$-$)$ Cauchy.

Proof. Since f is Lipschitz continuous,

$$d_Y(f(x_n), f(x_m)) \le C_f\, d_X(x_n, x_m) \quad \forall n,m \in \mathbb{N} \Rightarrow$$

$$\lim_{\min(n,m)\to+\infty} d_Y(f(x_n), f(x_m)) \le C_f \lim_{\min(n,m)\to+\infty} d_X(x_n, x_m) = 0 \Rightarrow$$

(since (x_n) is Cauchy)

$$\lim_{\min(n,m)\to+\infty} d_Y(f(x_n), f(x_m)) = 0,$$ hence $(f(x_n))$ is $(d_Y$-$)$ Cauchy. $\square$

Corollary. "Usual" continuity does not imply Lipschitz-continuity.

Proof. "Usual continuity" does not generally preserve Cauchyness but due to the previous Lemma, Lipschitz continuity preserves Cauchyness. $\square$

Hence due to the previous Lipschitz-continuity is genuinely stronger than usual continuity

Examples:

A. (X,d) a m.s., with $(\mathbb{R}, d_u)$. Let $z \in X$. Define

$$f_z : X \to \mathbb{R}, \text{ as } f_z(x) := d(z,x)$$

Let $x,y \in X$, $d_u(f_z(x), f_z(y)) = |f_z(x) - f_z(y)| =$

$$= |d(z,x) - d(z,y)| \leq d(x,y)$$

dual version of triangle inequality

Hence we have shown that

$$\forall x,y \in X \quad d_u(f_z(x), f_z(y)) \leq d(x,y)$$

hence f_z is d_u/d - Lipschitz continuous with Lipschitz coefficient equal to 1, and this holds $\forall z \in X$.

– Generally Metrics are Lipschitz continuous!

B. $X = \mathbb{R}^q$, $d_X = d_I$, $Y = \mathbb{R}^p$, $d_Y = d_I$

in $\mathbb{R}^q$ $\qquad$ $\mathbb{R}^p$

Consider the following: α. Euclidean Norm in $\mathbb{R}^q$

$$x \in \mathbb{R}^q, \quad \|x\| := \left(\sum_{i=1}^{q} x_i^2\right)^{1/2}$$

$$x = (x_1, x_2, \dots, x_q)$$

For $x,y \in \mathbb{R}^q$: $d_I(x,y) = \|x-y\|$

b. The Euclidean norm in $\mathbb{R}^p$ is analogous.

c. Let A be a $p \times q$ real matrix, i.e. $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1q} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2q} \\ \vdots & & \ddots & \\ \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pq} \end{pmatrix}$ then the Frobenius norm of A is $\|A\|_F = \left(\sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_{ij}^2 \right)^{1/2}$.

d. $x \in \mathbb{R}^q \Rightarrow \underset{pq\ q\times 1}{Ax} \in \mathbb{R}^p$

It is possible to prove that $\|\cdot\|$ and $\| \|_F$ satisfy the following sub-multiplicative property:

$$\underbrace{\|Ax\|}_{\text{Euclidean norm in } \mathbb{R}^p} \leq \|A\|_F \underbrace{\|x\|}_{\text{Euclidean norm in } \mathbb{R}^q}$$

End of lecture 10

e. Let $f: \mathbb{R}^q \to \mathbb{R}^p$ that is everywhere differentiable in $\mathbb{R}^q$, which implies that for any $x \in \mathbb{R}^q$, the Jacobian $\dfrac{\partial f(x)}{\partial x'}$ (i.e. the matrix defined as $f(x) \in \mathbb{R}^p$ $\forall x \in$, $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_p(x) \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}$

$$\frac{\partial f}{\partial x'}(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x), & \frac{\partial f_1}{\partial x_2}(x), & \ldots, & \frac{\partial f_1}{\partial x_q}(x) \\ \frac{\partial f_2}{\partial x_1}(x), & \frac{\partial f_2}{\partial x_2}(x), & \ldots, & \frac{\partial f_2}{\partial x_q}(x) \\ \vdots & & & \\ \frac{\partial f_p}{\partial x_1}(x), & \frac{\partial f_p}{\partial x_2}(x), & \ldots, & \frac{\partial f_p}{\partial x_q}(x) \end{pmatrix}_{p \times q}$$

is well defined $\forall x \in \mathbb{R}^q$. This is equivalent to that $0 \le \| \frac{\partial f(x)}{\partial x'} \|_F < +\infty$ $\forall x \in \mathbb{R}^q$. (i.e. $x \to \| \frac{\partial f(x)}{\partial x'} \|_F$ is a well defined function $\mathbb{R}^q \to \mathbb{R}$). Iff this $x \to \| \frac{\partial f(x)}{\partial x'} \|_F$ mapping is bounded then f is termed everywhere differentiable with bounded derivative. (Remember that this is equivalent to that $\sup_{x \in \mathbb{R}^q} \| \frac{\partial f}{\partial x'}(x) \|_F < +\infty$).

Theorem. If $f : \mathbb{R}^q \to \mathbb{R}^p$ that is everywhere differentiable with bounded derivative, then f is $(d_I / d_I -)$

Euclidean Metric in $\mathbb{R}^p$ — Euclidean Metric in $\mathbb{R}^q$

Lipschitz continuous, with $C_f = \sup_{x \in \mathbb{R}^q} \| \frac{\partial f}{\partial x'}(x) \|_F$.

Proof. Let arbitrary $x, y \in \mathbb{R}^q$. We have that

$$d_I (f(x), f(y)) = \| f(x) - f(y) \|$$

Euclidean Metric in $\mathbb{R}^p$

Since f is everywhere differentiable we have due the Mean value theorem that

$(*)$ $f(x) = f(y) + \frac{\partial f}{\partial x'}(y^*)(x-y)$ with y^* some element of $\mathbb{R}^q$ that lies in the line that connects x and y (this line is well-defined since $\mathbb{R}^q$ is convex, and $\frac{\partial f}{\partial x'}(y^*)$ is well-defined since f is everywhere differentiable)

$$(*) \Leftrightarrow \underbrace{f(x) - f(y)}_{\in \mathbb{R}^p} = \underbrace{\frac{\partial f}{\partial x'}(y^*)(x-y)}_{\in \mathbb{R}^p} \Rightarrow$$

$$\|f(x) - f(y)\| = \|\underbrace{\frac{\partial f}{\partial x'}(y^*)}_{A}\underbrace{(x-y)}_{x}\| \underset{\text{submult.}}{\leq} \|\frac{\partial f}{\partial x'}(y^*)\|_F \underbrace{\|x-y\|}_{\text{Euclidean norm in } \mathbb{R}^q}$$

$$\leq \underline{\sup_{x \in \mathbb{R}^q} \|\frac{\partial f}{\partial x'}(x)\|} \|x-y\|$$

Since $\sup_{x \in \mathbb{R}^q} \|\frac{\partial f}{\partial x'}(x)\| < +\infty$ we obtain that for the well defined $C_f := \sup_{x \in \mathbb{R}^q} \|\frac{\partial f}{\partial x'}(x)\|$ (this is independent of x, y) we obtain

$$\|f(x) - f(y)\| \leq C_f \|x-y\| \Leftrightarrow$$

$$\underbrace{d_I(f(x), f(y))}_{\text{in } \mathbb{R}^p} \leq C_f \underbrace{d_I(x,y)}_{\text{in } \mathbb{R}^q}$$

and since x, y are arbitrary and C_f is independent of them we have proven that:

$$\forall x,y \in \mathbb{R}^q, \quad d_I(f(x), f(y)) \leq C_f \, d_I(x,y)$$

and the result follows. $\square$

Remark. There is a partial converse to the previous theorem. If $f: \mathbb{R}^q \to \mathbb{R}^p$ is $\left(d_I/d_I -\right)$ Lipschitz continuous then f is almost everywhere differentiable with bounded derivative (where defined). The term "almost" means that there can exist $x \in \mathbb{R}^q$ at which f is not differentiable that form a negligible subset of $\mathbb{R}^q$.

Remark. More generally the previous theorem holds whenever $f: A \to \mathbb{R}^p$, $A \subseteq \mathbb{R}^q$, f is everywhere differentiable in A with bounded derivative, and $\forall x,y \in A$ there exists a line in A that connects them (e.g. A is convex).

Example: $p = q = 1$, $d_I = d_u$, $A = (-1,1)$ $f: (-1,1) \to \mathbb{R}$

convex

$f(x) = e^x$ (the exponential function restricted to $(-1,1)$).

$$\frac{df}{dx} = e^x \; \forall x \in (-1,1), \quad \sup_{x \in (-1,1)} \left\| \frac{\partial f}{\partial x}(x) \right\|_I = \sup_{x \in (-1,1)} |e^x| = \sup_{x \in (-1,1)} e^x$$

$= e^1 = e < +\infty$. Hence the exponential function restricted to $(-1,1)$ is Lipschitz continuous with $C_f = e$.

Exe. Is $f(x) = e^x : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous?

Compositional Property of Lipschitz Continuity.

(X, d_X), (Y, d_Y), (Z, d_Z). $X \xrightarrow{f} Y$ $\left(d_Y/d_X - \right)$ Lipshitz continuous, $Y \xrightarrow{g} Z$ $\left(d_Z/d_Y - \right)$ Lipshitz Continuous.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] \arrow[d, dash] & Y \arrow[r, "g"] & Z \arrow[d, dash] \\ d_X & & d_Z \end{tikzcd}$$

Is $g \circ f$ $\left(d_Z/d_X - \right)$ Lipschitz-Continuous?

Let $x, y \in X$, then $d_Z(g(\underbrace{f(x)}_{\in Y}), g(\underbrace{f(y)}_{\in Y})) \leq$ $\exists\, C_g > 0$ Lipsh. coef. of g

$C_g \underline{d_Y(f(x), f(y))} \leq C_g \cdot C_f \, d_X(x,y)$ $\exists\, C_f > 0$ Lipsh. Coef. of f

Since x, y are arbitrary we have that:

$$\forall x, y \in X \quad d_Z(g\circ f(x), g\circ f(y)) \leq C_g C_f \, d_X(x,y)$$

Hence $g \circ f$ is d_Z/d_X - Lipschitz continuous and $C_{g\circ f} = C_g \cdot C_f$.

* extends to any finite number of compositions (prove!)