# Lecture 1

- Syllabus [plus some procedural info]
- Preliminaries [Mathematical]
- Metrics and Metric Spaces
[Initial Definitions and Examples]

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# Syllabus of the Course: “Mathematical Economics”

**General Information:**
Office: Derigni Wing, 4th floor.[1] essentially not applicable
e-Class: https://eclass.aueb.gr/courses/OIK231/.[2]
Microsoft Teams Code: hvf14gm[3]
Tutor: Dimitrios Zaverdas, Office: Derigni Wing, 5th floor, E-Mail: zaverdasd@aueb.gr
Tutorial Information: tba. will not start before 3rd lecture at the earliest

**Course Description**
The course is an introduction to notions of mathematical analysis appearing in the theory of metric spaces with applications in economic theory and/or econometrics. We examine topological notions enabled in general metric spaces. Examples are the notion of *convergence* of sequences of elements, or the *continuity* of functions defined between them, finitary notions such as compactness, etc.
We are also occupied with non-topological notions, such as uniformities and completeness, and their interplay with the topological ones.
In this respect we construct a vocabulary which initially enables us to address the issue of approximation of optimisation problems and possibly consider relevant applications. Furthermore the aforementioned construction enables us to state and prove a variety of fixed point theorems. We use them in order to establish *existence (and occasionally uniqueness and/or approximability)* of solutions of general systems of equations. We apply those notions to problems appearing in dynamic optimisation, game theory, etc.
The combination of the aforementioned applications enables the unified consideration of both the existence of solutions in problems appearing in economic theory as well as the approximation of those (potentially not easily tractable) solutions with ones that are possibly easier to derive.

[1]Due to the pandemic any communication with the course’s instructor and/or the tutor will be exclusively held electronically.

[2] The course's e-class contains the course’s blog, notes, exercises, further readings and information concerning the lectures, corrections, announcements, etc. The relevant material could be updated during the course. The students must consult the e-class systematically and are strongly encouraged to upload questions, answers, comments, etc.

[3]Due to the pandemic the course’s lectures and tutorials will be exclusively held electronically and via the particular MS Teams group.

**Outline**

The following consists of a synopsis of the course material. It is understood that any partial modification, rearrangement, etc, is in the instructor's facility.

**A.1** Sets, cardinality, sequences, structures and morphisms. Metric and metrizable spaces, metrics, open and closed balls. Topological notions enabled by the existence of open balls: open and closed subsets, convergence and separation properties, continuity and characterization by convergent sequences, continuous mapping theorem. Compactness and connectedness. Non topological notions: bounded and totally bounded metric spaces, metric entropy, metric completeness, Cauchy sequences, Lipchitz and uniform continuity. Topological comparison and general comparison between metrics defined on the same set. Examples: discrete spaces, Euclidean spaces, spaces of bounded and continuous functions, etc. Self-maps, contractions and the Banach fixed point theorem. Approximation of the unique fixed point. Generalisations and the fixed point theorem of Matkowski. Retractions, the Lemma of Borsuk and Ulam, the Brower's fixed point theorem and generalisations.

**A.2** Applications: Convergence of sequences of minimisers and approximation of optimisation problems. Parametric optimisation. Existence and uniqueness of solutions to functional equations such as the Bellman equation in dynamic programming, or the Fredholm integral equation. Differential equations and Picard's Theorem. Existence of Nash equilibria in games. Sequences of games, notions of limit games and convergence of sequences of Nash equilibria.

**B.** (If time permits): Probability theory on metric spaces. Borel algebras and measures. Measurability and random elements with values on metric spaces. Polish spaces, measurability of suprema, the argmax theorem and consistency of M-estimators. Examples.

**Indicative Readings**

The following references are merely indicative. During the lectures this catalogue can be enriched with further readings. In any case the students are *strongly advised to study from more available sources and try to solve plethora of exercises*.


1. Aliprantis Ch., and K.C. Border. *Infinite Dimensional Analysis.* Springer, 2005.
2. Ok Efe. *Real Analysis with Economic Applications.* Princeton University Press, 2007.
3. Corbae D., Stinchcombe M, and J. Zeman. *An Introduction to Mathematical Analysis for Economic Theory and Econometrics.* Princeton U.P., 2009.
4. O'Searcoid, M. *Metric Spaces*. Springer Science & Business Media, 2006.
5. Sutherland, Wilson Alexander. *Introduction to metric and topological spaces*. Oxford University Press, 1975.
6. Border, K. C. *Fixed Point Theorems with Applications to Economics and Game Theory*. *Cambridge Books,* 1990.
7. Ambrosio, Luigi, and Paolo Tilli. *Topics on analysis in metric spaces*. Vol. 25. Oxford University Press on Demand, 2004.
8. Subrahmanyam, P. V. *Elementary Fixed Point Theorems.* Springer, 2018.

# A.

## Procedural Info:

available at MS Teams right after each lecture

* Eclass dep:

Notes
already there
could be updated

Lectures Whiteboards
those notes will be concurrently uploaded
(Most of lectures 2019-20 already there)

Tutorial Notes
already there
could be updated

Some Bibliography
links etc
could be updated

Exercises
previous years
could be updated

Optional Exercises
will be updated

Blog-roll
lectures reviews

etc (e.g. announcements)

* MS Teams dep: Analogous info to the above; Could also contain video lectures and tutorials, complementary stuff, etc.

* Complementary Evaluation: Optional Exercises
→ Available around May, could partially extend teaching material [Upload details tba]

* In any case: Try to solve as many exercises as possible

* We are not using a particular textbook: try to approximate the network of notions by as many sources as possible [Notes + Whiteboards etc provide with the main skeleton]

* Course's Aim:

Construct a language of analysis in Metric Spaces

Autonomous Value

View towards FPT with Applications

# B. Some Preliminary Notions

quick overview of

Not exhaustive

any other needed will be locally introduced

Analysis on spaces with a notion of distance

Some auxiliary ideas - especially related to $\mathbb{R}$ will be useful:

— Bounded subsets of $\mathbb{R}$

* $A \subseteq \mathbb{R}$ bounded from above iff $\exists M \in \mathbb{R}: x \leq M, \forall x \in A$

if and only if

$\forall M \in \mathbb{R}, \exists x \in A: x > M$

$M$ is termed upper bound of $A$ - Not unique (why?)

the smallest upper bound $\sup_{x \in A} x$ exists iff $A$ upper bounded and it is unique

* Dually $A \subseteq \mathbb{R}$ bounded from below iff $\exists m \in \mathbb{R}: m \leq x \ \forall x \in A$, $m$ is termed lower bound

Non unique, the greatest upper bound, $\inf_{x \in A} x$

exists iff $A$ is lower bounded and is unique

* $A$ is bounded iff it is upper and lower bounded $\iff \exists\, m, M \in \mathbb{R} : m \leq x \leq M \ \forall x \in A$

$$\iff \exists \inf_{x\in A} x \text{ and } \sup_{x \in A} x$$

$$\iff \exists\, M > 0 : \ |x| \leq M \ \forall x \in A$$

show it! absolute bound [could facilitate computations]

$$\iff \sup_{x\in A} |x| \text{ exists } (< +\infty)$$

E.g.

i. $A = (-\infty, 1)$ upper bounded, $\sup A = 1$, not lower bounded

ii. $A = [1, +\infty)$ lower bounded, $\inf A = 1$, not upper bounded

iii. if $A$ has a min $\Rightarrow$ lower bounded, $\inf_{x\in A} x = \min_{x \in A} x$ - Dually if $A$ has a max $\Rightarrow$ upper bounded and $\sup_{x\in A} x = \max_{x\in A} x$

$\Rightarrow$ iii. if $A$ is finite $\Rightarrow$ bounded

ii. $A = \mathbb{N}$ not bounded : Suppose not

$\Rightarrow \exists\, M > 0 : \quad |n| \leq M \ \forall n \in \mathbb{N} \iff n \leq M \ \forall n \in \mathbb{N}$

impossible

$\rightarrow$ Hence $A$ is not bounded iff

$\forall M > 0 \quad \exists\, x \in A : \quad |x| > M. \ \square$

— Bounded Real functions :

* $X$ a non-empty set, $f : X \to \mathbb{R}$ a real function defined on $X$.

ex. Define Pointwise Multiplication

* If $f, g : X \to \mathbb{R}$, $(f+g) : X \to \mathbb{R}$ where

$(f+g)(x) := f(x) + g(x)$ [Pointwise addition]

* If $\lambda \in \mathbb{R}$ $(\lambda f) : X \to \mathbb{R}$ where

$(\lambda f)(x) := \lambda f(x)$ [Scalar Multiplication]

$\subseteq \mathbb{R}$

* $f: X \to \mathbb{R}$ is bounded iff $f(X) := \{f(x), x \in X\}$ is a bounded subset of $\mathbb{R}$ $\iff$ $\boxed{\sup_{x\in X} |f(x)| < +\infty}$

* $B(X,\mathbb{R}) = \{f : X \to \mathbb{R},\ f \text{ bounded}\}$

bounded

↳ set of bounded real functions defined on $X$

- $B(X,\mathbb{R}) \neq \emptyset$ since it contains the constant functions

e.g. $f: X \to \mathbb{R}$ with $f(x) = 0 \ \forall x \in X$ is bounded since $\sup_{x\in X} |f(x)| = \sup_{x \in X} |0| = 0$ why?

- if $f(x) \leq g(x)\ \forall x \in X$ $\sup f \leq \sup g$
- $\sup(f+g) \leq \sup f + \sup g$
- $\sup|\lambda f| \leq |\lambda| \sup|f|$

→ Show it

If $f,g \in B(X,\mathbb{R}) \Rightarrow f+g \in B(X,\mathbb{R})$

tri.ineq $|\alpha+\beta| \leq |\alpha|+|\beta|$

since $\sup_{x\in X} |(f+g)(x)| = \sup_{x \in X} |f(x)+g(x)| \leq \sup_{x\in X}(|f(x)|+|g(x)|) \leq \sup_{x\in X}|f(x)| + \sup_{x\in X}|g(x)| < +\infty$

$<+\infty$ $<+\infty$

Show that if $f \in B(X,\mathbb{R})$, $\lambda \in \mathbb{R}$ $\lambda f \in B(X,\mathbb{R})$ $\square$

- Reminder: Cartesian Product

$X, Y \neq \phi$, $X \times Y = \{ (x,y), x \in X, y \in Y \}$

↳ ordered pairs of elements of $X$, $Y$

e.g. $X = \{\alpha\}$
$Y = \{0,1\}$

$\{\alpha\} \times \{0,1\} = \{(\alpha,0),(\alpha,1)\}$

# C. Introduction to Metric Spaces

Metric Space: A set coupled with mathematical structure that attributes a notion of distance between every pair of elements

↳ Distance function or Metric

**Definition:** Suppose that $X \neq \phi$. A function
* $d: \underbrace{X \times X}_{\text{pairs of elements}} \to \mathbb{R}$ (real function) is called a Metric

or a distance function on $X$ iff it satisfies:

① $d(x,y) \geq 0 \quad \forall x,y \in X$ [$\Leftrightarrow (x,y) \in X \times X$]

[positive definite]

② $d(x,y) = 0 \Longleftrightarrow x = y$ [separation]

③ $d(x,y) = d(y,x) \quad \forall x,y \in X$ [symmetry]

④ $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$

[triangle inequality]

$\square$

Remarks: - 1-4 codify our geometric intuition of "measuring distances"

- When $d$ satisfies 1,3,4 and $\forall x \in X$, $d(x,x) = 0$ [partially 2] then it is termed a pseudo-distance (pseudo-metric)

↳ does not necessarily separate points

↳ every distance is a pseudo-distance

Definition. The pair $(X,d)$ is called a Metric space [Pseudo-Metric when d is a pseudo-distance]

* $(X, d)$ → Codifies the info of a set where distances between its points can be attributed

carrier    Metric

# Examples:

## 1. Discrete Spaces

X is anything (as long as it is non empty)

$d_1 : X \times X \to \mathbb{R}$ defined by

$x, y \in X$, $d_1(x,y) := \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases}$

We have to check that it is well defined - obviously $d(x,y) \in \mathbb{R}$ $\forall x, y \in X$

① Obvious, ② obvious,

③ Let $x,y \in X$, $d(y,x) = \begin{cases} 0, & y=x \\ 1, & y \neq x \end{cases}$

$= \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases} = d(x,y)$

④ Let $x,y,z \in X$ then Enumerating cases

$d_1(x,z) + d_1(z,y) = \begin{cases} {}^{a}\,0 & \text{iff } x=z \text{ \& } z=y \\ {}^{b}\,1 & \text{iff } x=z \text{ and } z \neq y \\ & \text{or } x \neq z \text{ and } z=y \\ {}^{c}\,2 & \text{iff } x \neq z \text{ \& } z \neq y \end{cases}$

$\alpha \Rightarrow d(x,y)=0$ $[x=z \text{ \& } z=y \Rightarrow x=y,\ d_1(x,y)=0]$

$b,c$ in any case $d(x,y) \leq 1$

$d_1$ is termed a discrete metric

$(X, d_1)$ is termed a discrete space (discrete metric space)

↳ interesting properties to follow!

* Since X is non empty arbitrary EVERY non empty SET can be endowed with $d_L$ and become a discrete space.

* The "set" of metric spaces is non empty hence can become a discrete space → Russell's paradox – it cannot be a set [Category]

— Suppose for now that $X = \mathbb{R}$, obviously $\mathbb{R}$ can be endowed with $d_L$ and become discrete but not only:

usual distance

2. $X = \mathbb{R}$ $\quad \underbrace{d_a(x,y) = |x-y|}$ , $x, y \in \mathbb{R}$

the properties hold trivially

$(\mathbb{R}, d_a)$ is the real line endowed with the usual distance

3. Let $f: \mathbb{R} \to \mathbb{R}$ be 1-1 $(f(x)=f(y) \Leftarrow\Rightarrow x=y)$

Consider $X=\mathbb{R}$, $d_f(x,y) := |f(x)-f(y)|$, $\forall x,y \in \mathbb{R}$

$d_f$ is a metric:

(1) $\forall x,y \in \mathbb{R}$ $d_f(x,y) = |f(x)-f(y)| \geq 0$

(2) $0 = d_f(x,y) = |f(x)-f(y)| \Leftarrow\Rightarrow$

$f(x)=f(y) \overset{1-1}{\Leftarrow\Rightarrow} x=y$

(3) $\forall x,y \in \mathbb{R}$ $d_f(y,x) = |f(y)-f(x)| = |f(x)-f(y)| = d_f(x,y)$

(4) $\forall x,y,z \in \mathbb{R}$, $d_f(x,y) = |f(x)-f(y)| =$

$|\alpha+\beta| \leq |\alpha|+|\beta|$
$\alpha = f(x)-f(z)$
$\beta = f(z)-f(y)$

$= |(f(x)-f(z)) + (f(z)-f(y))| \leq |f(x)-f(z)|$

$+ |f(z)-f(y)| = d_f(x,z) + d_f(z,y)$

e.g. $f(x) = e^x$ [Is it 1-1?]

$d_e(x,y) = |e^x - e^y|$

Properties 1, 3, 4 and part of 2 hold even if $f$ is not 1-1

e.g. $f(x) = x$ [It is 1-1] $d_f(x,y) = |f(x)-f(y)| =$

$= |x-y| = d_x(x,y)$

* if $f$ is not 1-1 the $d_f$ is merely a Pseudo-Metric - why?

* $(\mathbb{R}, d_1) \neq (\mathbb{R}, d_e) \neq (\mathbb{R}, d_\alpha)$

Different Metric Spaces

[Can have quite different properties]

→ It is possible that the same carrier can be endowed with different (pseudo-) metrics resulting into different spaces - with different properties [Different d's shed light to different "relations" between elements of X]

→ It is also possible that different metrics can be related to each other resulting to

## Correlated Properties

- We will examine more complicated examples and relations in the next lecture. For now:

  * $d_f = d_\alpha$ for $f(x) = x$

  * $X$ general, $c \in \mathbb{R}$

$$d_c(x,y) := \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases}$$

For which values of $c$ is $d_c$ a pseudo-metric, a metric? Are there relations between $d_c$ and $d_1$? [Exe!]

$1 = d_*(0,0) > \frac{1}{3} + \frac{1}{3}$

$= d_*(0,\frac{1}{3}) + d_*(\frac{1}{3},0)$

$d_*$ does not respect the triangle inequality

Counterexample $X = \mathbb{R}$

$d_* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$x, y \in \mathbb{R}$

$$d_*(x,y) := \begin{cases} |x-y|, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$x = y = 0$

$z = 1/3$

$d_*(0,0) = 1$

$d_*(0,1/3) = d_*(1/3,0)$

$= 1/3$

$d_*(0,0) = 1 \Rightarrow$ (2) does not hold.