Spurious Regression

We have seen that the asymptotic behavior of the OLS estimator in the context of an AR(1) model, changes dramatically when $(y_t)_{t\in\mathbb{N}}$ is a random walk process. It is then reasonable to expect that the OLS asymptotics would also be quite different if both the dependent and the independent variables of a regression model, are random walk processes.

Consider two independent random walk processes $(y_t)_{t\in\mathbb{N}}$ and $(x_t)_{t\in\mathbb{N}}$ as a solution of the relevant unit root recursions:

$$y_t = a_{1_0} y_{t-1} + u_t \quad , \quad t \in \mathbb{N} \quad , \quad (u_t)_{t\in\mathbb{N}} \quad , \quad a_{1_0} = 1 \quad , \quad y_0 = 0$$

and

$$x_t = a_{2_0} x_{t-1} + v_t \quad , \quad t \in \mathbb{N} \quad , \quad (v_t)_{t\in\mathbb{N}} \quad , \quad a_{2_0} = 1 \quad , \quad x_0 = 0$$

where $(u_t)_{t\in\mathbb{N}}$ and $(v_t)_{t\in\mathbb{N}}$ are independent, strictly stationary and ergodic, s.m.d with respect to some filtration, processes say:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} \sim \text{iid } \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} \right)$$

Thereby, y_t with all leads and lags, is independent of x_t with all leads and lags.

Now, consider the OLS regression of y_t on x_t:

$$Y = b_0 X + \epsilon$$

such that $Y = (y_t)_{t=1}^{T}$, $X = (x_t)_{t=1}^{T}$.

We would expect that the OLSE of b_0, b_T should tend to 0 as $T \to \infty$, since (y_t) and (x_t) are uncorrelated with each other by construction.

However, the above regression will have a tendancy to "discover" a relationship between y_t and x_t despite the fact that there is none. This in known as the spurious regression phenomenon.

The OLS estimator of b_0 is denoted by $b_T = \dfrac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{T} x_t^2}$.

For operational reasons, in order to examine the asymptotic behavior of the OLSE, we can consider $b_{T-1} = \dfrac{\sum_{t=1}^{T-1} x_t y_t}{\sum_{t=1}^{T-1} x_t^2}$, which asymptotically will converge to the same limit as b_T.

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Note that,

$$\sum_{t=1}^{T-1} x_t y_t \overset{t^*=t+1}{=\!=\!=} \sum_{t^*=2}^{T} x_{t^*-1}\, y_{t^*-1} \underset{y_0=0}{\overset{x_0=0}{=\!=\!=}} \sum_{t^*=1}^{T} x_{t^*-1}\, y_{t^*-1}.$$

and

$$\sum_{t=1}^{T-1} x_t^2 \overset{t^*=t+1}{=\!=\!=} \sum_{t^*=2}^{T} x^2_{t^*-1} \overset{x_0=0}{=\!=\!=} \sum_{t^*=1}^{T} x^2_{t^*-1}.$$

Given the process $(u_t)_{t\in\mathbb{N}}$ and $T \geq 1$, $r \in [0,1]$ we can define the partial sum process of $(u_t)_{t\in\mathbb{N}}$ over $[0,1]$, $W_{Y_T}(\cdot)$ where $W_{Y_T}(r) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor rT \rfloor} u_i$.

For $1 \leq t \leq T$ and $t-1 \leq rT < t$:

$$W_{Y_T}(r) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor rT \rfloor} u_i = \frac{1}{\sqrt{T}} \sum_{i=1}^{t-1} u_i = W_{Y_T}\left(\frac{t-1}{T}\right)$$

But when $r = \frac{t}{T}$:

$$W_{Y_T}(r) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor rT \rfloor} u_i = \frac{1}{\sqrt{T}} \sum_{i=1}^{t} u_i = W_{Y_T}\left(\frac{t}{T}\right)$$

Hence, $\forall\, \omega \in \Omega$, $W_{Y_T}(\cdot, \omega)$ is right continuous with finite left limit.

Then,

$$\int_0^1 W_{Y_T}^2(r)\,dr = \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} W_{Y_T}^2(r)\,dr = \sum_{t=1}^{T} W_{Y_T}^2\left(\frac{t-1}{T}\right) \int_{\frac{t-1}{T}}^{\frac{t}{T}} dr = \sum_{t=1}^{T} W_{Y_T}^2\left(\frac{t-1}{T}\right) \frac{1}{T}$$

$$= \frac{1}{T} \sum_{t=1}^{T} W_{Y_X}^2\left(\frac{t-1}{T}\right) = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor \frac{t-1}{T} T \rfloor} u_i \right)^2 = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{T} \left(\sum_{i=1}^{t-1} u_i \right)^2$$

$$= \frac{1}{T^2} \sum_{t=1}^{T} \left(\sum_{i=1}^{t-1} u_i \right)^2$$

Similarly, given the process $(v_t)_{t\in\mathbb{N}}$ and $T \geq 1$, $r \in [0,1]$, we can define the partial sum process of $(v_t)_{t\in\mathbb{N}}$ over $[0,1]$, $W_{X_T}(\cdot)$ where $W_{X_T}(r) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor rT \rfloor} v_i$.

For $1 \leq t \leq T$ and $t-1 \leq rT < t$, $W_{Y_T}(r) = 1/\sqrt{T} \sum_{i=1}^{t-1} v_i = W_{Y_T}\left(\frac{t-1}{T}\right)$

But when $r = t/T$, $W_{Y_T}(r) = 1/\sqrt{T} \sum_{i=1}^{t} v_i = W_{Y_T}(t/T)$.

Hence, $\forall\, \omega \in \Omega$, $W_{X_T}(\cdot, \omega)$ is right continuous with finite left limit. Then,

$$\int_0^1 W_{X_T}^2(r)\,dr = \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{t/T} W_{X_T}^2(r)\,dr = \sum_{t=1}^{T} W_{X_T}^2\left(\frac{t-1}{T}\right) \int_{\frac{t-1}{T}}^{\frac{t}{T}} dr = \frac{1}{T^2} \sum_{t=1}^{T} \left(\sum_{i=1}^{t-1} v_i \right)^2.$$

Also,

$$\int_0^1 W_{X_T}(r)\, W_{Y_T}(r)\,dr = \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} W_{X_T}(r)\, W_{Y_T}(r)\,dr = \sum_{t=1}^{T} W_{X_T}\left(\frac{t-1}{T}\right) W_{Y_T}\left(\frac{t-1}{T}\right) \int_{\frac{t-1}{T}}^{\frac{t}{T}} dr =$$

$$\frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor \frac{t-1}{T} T \rfloor} v_i \right) \left(\frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor \frac{t-1}{T} T \rfloor} u_i \right) = \frac{1}{T^2} \sum_{t=1}^{T} \left(\sum_{i=1}^{t-1} v_i \right) \left(\sum_{i=1}^{t-1} u_i \right)$$

So, if we consider the vector process $(z_t)_{t\in\mathbb{N}}$, $z_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$ where $(u_t)_{t\in\mathbb{N}}$ and $(v_t)_{t\in\mathbb{Z}}$ are independent, we can construct,

$$W_T(r) = \begin{pmatrix} W_{Y_T}(r) \\ W_{X_T}(r) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{T}} \sum_{i=1}^{[rT]} u_i \\ \frac{1}{\sqrt{T}} \sum_{i=1}^{[rT]} v_i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{T}} \sum_{i=1}^{t-1} u_i \\ \frac{1}{\sqrt{T}} \sum_{i=1}^{t-1} v_i \end{pmatrix} = W_T\left(\frac{t-1}{T}\right), \ \forall\ r \in \left[\frac{t-1}{T}, \frac{t}{T}\right]$$

and $W_T(r) = W\left(\frac{t}{T}\right)$, for $r = \frac{t}{T}$.

Given the way we defined the partial sum processes and thereby the vector of partial sum processes, from the integral properties,

$$\int_0^1 W_T(r)\,W_T(r)'\,dr = \int_0^1 \begin{pmatrix} W_{Y_T}(r) \\ W_{X_T}(r) \end{pmatrix} \begin{pmatrix} W_{Y_T}(r) & W_{X_T}(r) \end{pmatrix} dr =$$

$$\int_0^1 \begin{bmatrix} W_{Y_T}^2(r) & W_{Y_T}(r)\,W_{X_T}(r) \\ W_{X_T}(r)\,W_{Y_T}(r) & W_{X_T}^2(r) \end{bmatrix} dr = \begin{bmatrix} \int_0^1 W_{Y_T}^2(r)\,dr & \int_0^1 W_{Y_T}(r)\,W_{X_T}(r)\,dr \\ \int_0^1 W_{X_T}(r)\,W_{Y_T}(r)\,dr & \int_0^1 W_{X_T}^2(r)\,dr \end{bmatrix} =$$

$$\frac{1}{T^2} \sum_{t=1}^{T} \begin{bmatrix} \left(\sum_{i=1}^{t-1} u_i\right)^2 & \left(\sum_{i=1}^{t-1} u_i\right)\left(\sum_{t=1}^{t-1} v_i\right) \\ \left(\sum_{i=1}^{t-1} v_i\right)\left(\sum_{i=1}^{t-1} u_i\right) & \left(\sum_{i=1}^{t-1} v_i\right)^2 \end{bmatrix}$$

Since $(z_t)_{t\in\mathbb{Z}}$ is a strictly stationary and ergodic, s.m.d. (w.r.t. some filtration) vector process, according to the multivariate Functional Central Limit Thm,

$$\Sigma^{-1/2}\,W_T(r) = \Sigma^{-1/2} \begin{pmatrix} W_{Y_T}(r) \\ W_{X_T}(r) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} W_Y(r) \\ W_X(r) \end{pmatrix} = W(r) \quad \text{as } T\to\infty$$

where $\Sigma = \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}$.

Thereby, using the (Functional) Continuous Mapping Thm:

$$\int_0^1 W_T(r)\,W_T(r)'\,dr \xrightarrow{d} \int_0^1 \Sigma^{1/2}\,W(r)\,W(r)'\,\Sigma^{1/2}\,dr =$$

$$\begin{bmatrix} \sigma_u^2 \int_0^1 W_Y^2(r)\,dr & \sigma_u\sigma_v \int_0^1 W_Y(r)\,W_X(r)\,dr \\ \sigma_u\sigma_v \int_0^1 W_Y(r)\,W_X(r)\,dr & \sigma_v^2 \int_0^1 W_X^2(r)\,dr \end{bmatrix} \quad (*) \quad \text{as } T\to\infty.$$

In order to examine the asymptotic behavior of the OLSE, we are interested in the joint asymptotic distribution of $\left(\sum_{t=1}^{T-1} x_t y_t\right)$ and $\left(\sum_{t=1}^{T-1} x_t^2\right)$.

▷ $$\frac{1}{T^2}\sum_{t=1}^{T-1} x_t y_t = \frac{1}{T^2}\sum_{t^*=1}^{T} x_{t^*-1}\, y_{t^*-1} = \frac{1}{T^2}\sum_{t^*=1}^{T}\left(\sum_{i=1}^{t^*-1} v_i\right)\left(\sum_{i=1}^{t^*-1} u_i\right)$$

$$= \int_0^1 W_{X_T}(r)\, W_{Y_T}(r)\, dr$$

▷ $$\frac{1}{T^2}\sum_{t=1}^{T-1} x_t^2 = \frac{1}{T^2}\sum_{t^*=1}^{T} x_{t^*-1}^2 = \frac{1}{T^2}\sum_{t^*=1}^{T}\left(\sum_{i=1}^{t^*-1} v_i\right)^2 = \int_0^1 W_{X_T}^2(r)\, dr$$

Therefore, given $(*)$ and the CMT:

$$b_{T-1} = \frac{\frac{1}{T^2}\sum_{t=1}^{T-1} x_t y_t}{\frac{1}{T^2}\sum_{t=1}^{T-1} x_t^2} \xrightarrow{d} \frac{\sigma_u \sigma_v \int_0^1 W_x(r)\, W_y(r)\, dr}{\sigma_v^2 \int_0^1 W_x^2(r)\, dr} \quad \text{as } T\to\infty.$$

Hence, the OLSE does not converge in probability to 0. It converges, in distribution to a non-normal random variable.

Cointegration

As we saw, regressing two unit root processes can produce nonsense. So, does it make sense to regress two non-stationary processes, which in many time series are considered together and they form equilibrium relationships?

Yes, if there exists a linear combination of them which is stationary.

So, if we set $w_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix}$ which is a non-stationary vector process and define $b = \begin{pmatrix} 1 \\ -b_2 \end{pmatrix}$, we can consider the linear combination of y_t and x_t,

$b' w_t = \begin{pmatrix} 1 & -b_2 \end{pmatrix}\begin{pmatrix} y_t \\ x_t \end{pmatrix} = y_t - b_2 x_t = \gamma_t$. If there exists a vector b such that γ_t is stationary, then we say that x_t and y_t are cointegrated.

- The intuition behind cointegration is that $b'w_t$ forms somehow a long-run equilibrium. It cannot deviate too far from the equilibrium otherwise economic forces will operate to restore the equilibrium. So it implies a set of dynamic long-run equilibrium between the variables.

- Estimates of the cointegrating relationships are super constistent i.e. they converge at rate T rather than $\sqrt{T}$.