An example of a Gaussian process

Consider the process $(y_t)_{t\in\mathbb{Z}}$ defined by $y_t = u_t\cos(t) + v_t\sin(\lambda t)$, $t\in\mathbb{Z}$, $\lambda\in\{0,1\}$,
where $\begin{pmatrix}u_t\\ v_t\end{pmatrix} \sim N(0_{2\times1}, Id_{2\times2})$, $t\in\mathbb{Z}$ $(v_t)_{t\in\mathbb{Z}}$ and $(u_t)_{t\in\mathbb{Z}}$ are i.i.d. and
u_t is independent of v_p for any $t\neq p$.
Is the process Gaussian? Is it strongly and/or weakly stationary?

Remember that, if we consider the random variables $X_1, X_2, \dots, X_p$ with expectation $\mu_k = E(X_k)$ and covariances $\sigma_{jk} = Cov(X_j, X_k)$ and define $\mu = (\mu_1, \mu_2, \dots, \mu_p)'$, $\Sigma = (\sigma_{jk})$: covariance matrix for $X_1, X_2, \dots, X_p$ then by definition the random vector $\mathbb{X} = (X_1, X_2, \dots, X_p)'$ is p-dimensionally normally distributed if and only $a'X \sim N(a'\mu, a'\Sigma a)$ for all $a = (a_1, \dots, a_p)'$
Obviously, $X_k = 0\cdot X_1 + 0\cdot X_2 + \dots + 1\cdot X_k + \dots + 0\cdot X_p$ is normal i.e. all marginal distributions in a p-dimensional normal distribution are one-dimensional normal. However, the reverse is not true.

Therefore, given that $\begin{pmatrix}u_t\\ v_t\end{pmatrix} \sim N(0_{2\times1}, Id_{2\times2})$

$u_t \sim N(0,1)$ and $v_t \sim N(0,1)$

Also, $\cos(t)\, u_t \sim N(0, \cos^2(t))$ and $\sin(\lambda t)\, v_t \sim N(0, \sin^2(\lambda t))$
since $\cos(t)$ and $\sin(\lambda t)$ at every t are just constants and not functions of the random variables
[Remember that, we can prove that if $X \sim N(\mu, \sigma^2)$ then $aX + b \sim N(a\mu + b, a^2\sigma^2)$]

Now, if we consider the random variable $y_t = \cos(t)\, u_t + \sin(\lambda t)\, v_t$
[Remember that, there is a theorem according to which if $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ where X_1, X_2 are independent then $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$]
then $y_t \sim N(0, \cos^2(t) + \sin^2(\lambda t))$
Hence, for $\lambda = 1$: $y_t \sim N(0,1)$ due to the Pythagorean identity
$\lambda = 0$: $y_t \sim N(0, \cos^2(t))$

Recall that, a Gaussian $\mathbb{R}$-valued process over Θ is a stochastic process for which every finite dimensional distribution (fidi) is a normal distribution. Moreover, it is possible to prove that a Gaussian process is completely characterized by the mean function and the covariance function, as knowledge of these functions allows us to write up every fidi.

In order to prove that $(y_t)_{t\in\mathbb{Z}}$ is Gaussian process consider an arbitrary fidi for any $K \in \mathbb{N}$:

$$\mathbb{P}(y_{t_1}\in \cdot, y_{t_2}\in \cdot, \dots, y_{t_K}\in \cdot) \underset{\substack{\text{since}\\ u_t, v_t \text{ independent}}}{\overset{\text{indep.}}{=}} \mathbb{P}(y_{t_1}\in \cdot)\cdot \mathbb{P}(y_{t_2}\in \cdot) \cdots \mathbb{P}(y_{t_K}\in \cdot)$$

$$= \mathcal{N}(\mu_1, \sigma_1^2)\cdot \mathcal{N}(\mu_2, \sigma_2^2) \cdots \mathcal{N}(\mu_K, \sigma_K^2)$$

where $\mathcal{N}(\mu_i, \sigma_i^2)$ represents a Gaussian probability density function. Since the product of Gaussian probability density functions is also a Gaussian functions, the product above is equal to a Gaussian density function. Given that the aforementioned fidi is chosen arbitrarily, every fidi is a normal distribution, therefore the stochastic process $(y_t)_{t\in\mathbb{Z}}$ is Gaussian.

Strict Stationarity

Remember that, a stochastic process $(y_t)_{t\in\mathbb{Z}}$ is (strictly) stationary iff every fidi remains invariant wrt time shifts.

Since, we have proved that $(y_t)_{t\in\mathbb{Z}}$ is a Gaussian process and we know that such a process is completely characterized by its mean and covariance function, it is sufficient to prove that these two functions are independent of t in order to prove the strict stationarity of the process.

$$E(y_t) = E\left[\cos(t)\,u_t + \sin(\lambda t)\,v_t\right] = \cos(t)\,\cancelto{0}{E(u_t)} + \sin(\lambda t)\cdot \cancelto{0}{E(v_t)} = 0 \quad \text{independent of } t$$

$$\begin{aligned}Cov(y_t, y_{t-K}) &= E(y_t y_{t-K}) = E\Big[\left[\cos(t)u_t + \sin(\lambda t)v_t\right]\left[\cos(t-K)u_{t-K} + \sin(\lambda(t-K))\,v_{t-K}\right]\Big]\\ &= E\Big[\cos(t)\cos(t-K)\,u_t u_{t-K} + \cos(t)\sin(\lambda(t-K))\,u_t v_{t-K}\\ &\quad + \sin(\lambda t)\cos(t-K)\,v_t u_{t-K} + \sin(\lambda t)\sin(\lambda(t-K))\,v_t v_{t-K}\Big]\end{aligned}$$

$$\overset{v_t,\, u_p \text{ indep.}}{=} \cos(t)\cos(t-k)\,E(u_t u_{t-k}) + \sin(\lambda t)\sin(\lambda(t-k))\,E(v_t v_{t-k})$$

$$= \begin{cases} \cos^2(t)\,E(u_t^2) + \sin^2(t)\,E(v_t^2), & k=0 \\ 0, & k \geq 1. \end{cases} = \begin{cases} \cos^2(t) + \sin^2(\lambda t), & k=0 \\ 0, & k \geq 1. \end{cases}$$

For $\lambda = 0$: $\operatorname{Cov}(y_t, y_{t-k}) = \begin{cases} \cos^2(t) + \sin^2(0), & k=0 \\ 0, & k \geq 1 \end{cases} = \begin{cases} \cos^2(t), & k=0 \\ 0, & k \geq 1. \end{cases}$ depends on t

For $\lambda = 1$: $\operatorname{Cov}(y_t, y_{t-k}) = \begin{cases} \cos^2(t) + \sin^2(t), & k=0 \\ 0, & k \geq 1. \end{cases} = \begin{cases} 1, & k=0 \\ 0, & k=0. \end{cases}$ independent of t

Thus, for $\lambda = 1$, $(y_t)_{t\in\mathbb{Z}}$ is strictly stationary while for $\lambda = 0$, $(y_t)_{t\in\mathbb{Z}}$ is not strictly stationary since the covariance function depends on t.

Weak Stationarity

A stochastic process $(y_t)_{t\in\mathbb{Z}}$ is weakly stationary (or covariance stationary or second order stationary) iff it satisfies the following conditions:

1. $E(y_t) \in \mathbb{R}$ and is independent of t.
2. $\operatorname{Var}(y_t) < +\infty$ and is independent of t
3. $\operatorname{Cov}(y_t, y_{t-k})$ is independent of t (but may depend on k) $\forall\, k \in \mathbb{Z}$

Given that we have already found the mean and covariance function, it is easy to see that:

for $\lambda = 0$: $E(y_t) = 0 \quad \forall\, t \in \mathbb{Z}$

$V(y_t) = \cos^2(t)$ depends on t

$\operatorname{Cov}(y_t, y_{t-k}) = 0 \quad \forall\, t \in \mathbb{Z},\ k \neq 0$

Hence, $(y_t)_{t\in\mathbb{Z}}$ is not weakly stationary when $\lambda = 0$.

for $\lambda = 1$: $E(y_t) = 0 \quad \forall\, t \in \mathbb{Z}$

$V(y_t) = 1 \quad \forall\, t \in \mathbb{Z}$

$\operatorname{Cov}(y_t, y_{t-k}) = 0 \quad \forall\, t \in \mathbb{Z},\ k \neq 0$

so, $(y_t)_{t\in\mathbb{Z}}$ is weakly stationary when $\lambda = 1$.

An example of a linear long-memory process

Definition: A weakly stationary process $(x_t)_{t \in \mathbb{Z}}$ with autocovariance function γ_k is said to have long memory if $\sum_{k=0}^{\infty} |\gamma_k| = \infty$.

- Long memory implies that the autocovariances decay to zero so slowly that their sum does not converge
- From an empirical approach, the presence of long memory can be defined in terms of the persistence of observed autocovariances

We can represent long memory time series as linear processes with "special" coefficients.

Consider $\epsilon = (\epsilon_t)_{t \in \mathbb{Z}}$ a strictly stationary white noise process $(\epsilon \sim WN(E(\epsilon_0^2)))$ and a real sequence $(\psi_j)_{j \in \mathbb{N}}$.

$\forall\, t \in \mathbb{Z}$ we can define the random variable

$$y_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \quad \text{where} \quad \psi_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \quad \Gamma(\cdot) \text{ is a Gamma function}, \quad 0 < d < 1/2$$

- The parameter d is called the memory parameter of the underlying stationary process. Under long memory $0 < d < 1/2$. (If the underlying process has short memory then $d = 0$)

- We can show that the coefficients are square summable i.e. $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. (Remember that square summability is a weaker condition than (absolute) summability)

Using the standard approximation derived from Sheppard's formula, that for large j, $\Gamma(j+a)/\Gamma(j+b)$ is well approximated by j^{a-b} it follows that

$$\psi_j \sim \frac{j^{d-1}}{\Gamma(d)} \simeq A j^{d-1} \qquad \text{(for a given value of } d) \qquad (1)$$

as $j \to \infty$ and an appropriate constant A.

Also, from the theory of infinity series, it is known that $\sum_{j=0}^{\infty} j^{-s}$ converges for $s > 1$ but otherwise diverges.

Therefore, from (1) it follows that $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ when $0 < d < 1/2$, so the coefficients of the process are square summable (but not absolutely summable)

- We can prove that the linear process $y = (y_t)_{t \in \mathbb{Z}}$ is strictly stationary.

- Remember, for a linear process defined as $y_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$:
  - $E(y_t) = E(\epsilon_0) \sum_{j=0}^{\infty} \psi_j = 0 \quad \forall\, t \in \mathbb{Z}$
  - $Var(y_t) = E(\epsilon_0^2) \sum_{j=0}^{\infty} \psi_j^2 < +\infty \quad \forall\, t \in \mathbb{Z}$
  - $\gamma_k = Cov(y_t, y_{t-k}) = E(\epsilon_0^2) \sum_{j=0}^{\infty} \psi_j \psi_{j+k} = E(\epsilon_0^2) \dfrac{\Gamma(k+d)\Gamma(1-2d)}{\Gamma(k+1-d)\Gamma(1-d)\Gamma(d)}$

using a similar argument as previously,
$\gamma_k \sim B k^{2d-1}$ as $k \to \infty$ and an appropriate constant B.

Therefore, $\sum_{k=0}^{\infty} \gamma_k$ diverges when $0 < d < 1/2$ and $(y_t)_{t \in \mathbb{Z}}$ is said to have long memory.

For <u>example</u>: $d = 1/4$, $\gamma_k \simeq k^{-1/2}$

$$\text{thus } \sum_{k=0}^{\infty} \gamma_k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}} = \infty$$

↳ Hence, we described a linear stochastic process which has square summable coefficients <u>but</u> the sum of its autocovariances diverges.