Athens University of Economics and Business
Department of Economics

Postgraduate Program - Master's in Economic Theory
Course: Econometrics II
Prof: Stelios Arvanitis
TA: Alecos Papadopoulos

Semester: Spring 2017-2018

March 22, 2018

An example of a "Long-memory" process

Definition: A weakly stationary stochastic process $(x_t)_{t\in\mathbb{Z}}$ with autocovariance function γ_k is said to have "long-memory" if $\sum_{k=0}^{\infty}|\gamma_k| = \infty$.

An example.

Consider $(u_t)_{t\in\mathbb{Z}}$ a strictly stationary White Noise process $\text{WN}(\sigma_u^2)$, and a real sequence $(\theta_j)_{j\in\mathbb{N}}$. $\forall t\in\mathbb{Z}$ we initially define

$$y_t = \sum_{j=0}^{\infty}\theta_j u_{t-j}, \qquad \theta_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \qquad d>0,$$

where $\Gamma(\cdot)$ is the Gamma function. The parameter d is called the "memory parameter". **We will see that in order for y_t to be well defined, we will have to strengthen the restrictions on the value of d to $0<d<1/2$.**

The Gamma function satisfies the identity

$$\Gamma(j+1) = j\Gamma(j) \Rightarrow \ \theta_j = \frac{\Gamma(j+d)}{j\Gamma(j)\Gamma(d)}$$

The following exact relation holds between the Gamma function and the Beta function

$$\mathrm{B}(j,d) = \frac{\Gamma(j)\Gamma(d)}{\Gamma(j+d)} \Rightarrow \ \theta_j = \frac{1}{j\,\mathrm{B}(j,d)}$$

Using the Stirling approximation for "large" j and fixed/small d as is our case, we have

$$\mathrm{B}(j,d) \approx \Gamma(d)\, j^{-d} \Rightarrow \theta_j \approx \frac{1}{j\,\Gamma(d)\, j^{-d}} \Rightarrow \theta_j \approx \frac{1}{\Gamma(d)\, j^{1-d}}$$ while note that $\theta_0 = 1$ exactly.

Note also that $\Gamma(d)$ is bounded and does not depend on the index.

Convergence.
From the theory of infinite series we know that

$\sum_{j=1}^{\infty} \frac{1}{j^{1-d}}$ diverges since $1-d \le 1,\ \forall d > 0$

So $\sum_{j=0}^{\infty} |\theta_j| = \sum_{j=0}^{\infty} \theta_j \longrightarrow 1 + \frac{1}{\Gamma(d)} \sum_{j=1}^{\infty} \frac{1}{j^{1-d}} \to \infty$

And **the coefficient series is not absolutely summable.**

But consider

$$\sum_{j=0}^{\infty} \theta_j^2 \longrightarrow 1 + \frac{1}{\left(\Gamma(d)\right)^2} \sum_{j=1}^{\infty} \frac{1}{j^{2-2d}}$$

Whether this converges depends on the actual value of d.

If $0 < d < 1/2$ we obtain $2 - 2d > 2 - 2\frac{1}{2} = 1$, in which case $\sum_{j=0}^{\infty} \theta_j^2 < +\infty$.

Namely, **for $0 < d < 1/2$ we have square-summability of the coefficient series.**

Square-summability guarantees that $y_t = \sum_{j=0}^{\infty} \theta_j u_{t-j}$ "converges in mean-square" to some random variable (see Hamilton's book Appendix 3.A, pp 69-70).

So **we eventually impose $0 < d < 1/2$,** and we have square-summability but not absolute summability. Square-summability (and hence convergence), implies also that we can interchange the expected value (an integral) with the infinite summation.

Weak stationarity of y_t .

1. $E(y_t)=E\left(\sum_{j=0}^{\infty}\theta_j u_{t-j}\right)=\sum_{j=0}^{\infty}\theta_j E(u_{t-j})=E(u_0)\sum_{j=0}^{\infty}\theta_j = 0\cdot\sum_{j=0}^{\infty}\theta_j = 0$

2. $\text{Var}(y_t)=E\left(\sum_{j=0}^{\infty}\theta_j u_{t-j}\right)^2 = E\left[\left(\sum_{j=0}^{\infty}\theta_j u_{t-j}\right)\left(\sum_{j=0}^{\infty}\theta_j u_{t-j}\right)\right]$

$$=E\left[\sum_{j=0}^{\infty}\theta_j^2 u_{t-j}^2+\sum_{j\neq i}^{\infty}\theta_j\theta_i\, u_{t-j}u_{t-i}\right]=E\left[\sum_{j=0}^{\infty}\theta_j^2 u_{t-j}^2\right]+E\left[\sum_{j\neq i}^{\infty}\theta_j\theta_i\, u_{t-j}u_{t-i}\right]$$

$$=\sum_{j=0}^{\infty}\theta_j^2 E\left(u_{t-j}^2\right)+\sum_{j\neq i}^{\infty}\theta_j\theta_i\, E\left(u_{t-j}\right)E\left(u_{t-i}\right)=\sigma_u^2\sum_{j=0}^{\infty}\theta_j^2 + 0 < +\infty$$

...finite and not depending on the index.

We see that we need square summability in order to obtain a finite variance.

3. Autocovariance .

$$\gamma_k = \text{Cov}(y_t, y_{t-k}) = E(y_t y_{t-k}) - E(y_t)E(y_{t-k}) = E(y_t y_{t-k})$$

$$=E\left[\left(\sum_{j=0}^{\infty}\theta_j u_{t-j}\right)\left(\sum_{j=0}^{\infty}\theta_j u_{t-k-j}\right)\right]=E\left[\left(\sum_{j=0}^{k-1}\theta_j u_{t-j}+\sum_{j=k}^{\infty}\theta_j u_{t-j}\right)\left(\sum_{j=0}^{\infty}\theta_j u_{t-k-j}\right)\right]$$

$$=E\left[\left(\sum_{j=0}^{k-1}\theta_j u_{t-j}\right)\left(\sum_{j=0}^{\infty}\theta_j u_{t-k-j}\right)\right]+E\left[\left(\sum_{j=k}^{\infty}\theta_j u_{t-j}\right)\left(\sum_{j=0}^{\infty}\theta_j u_{t-k-j}\right)\right]$$

$$=0+E\left[\left(\sum_{j=k}^{\infty}\theta_j u_{t-j}\right)\left(\sum_{j=0}^{\infty}\theta_j u_{t-k-j}\right)\right]=E\left(\theta_k u_{t-k}+\theta_{k+1}u_{t-k-1}+\ldots\right)\left(\theta_0 u_{t-k}+\theta_1 u_{t-k-1}+\ldots\right)$$

Cross-products are zero so we are left with

$$\gamma_k = E\left(\theta_0\theta_k u_{t-k}^2+\theta_1\theta_{k+1}u_{t-k-1}^2+\ldots\right)=\theta_0\theta_k E\left(u_{t-k}^2\right)+\theta_1\theta_{k+1}E\left(u_{t-k-1}^2\right)+\ldots$$

$$\Rightarrow \gamma_k = \sigma_u^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+k} = \sigma_u^2 \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} \frac{\Gamma(j+k+d)}{\Gamma(j+k+1)\Gamma(d)}$$

This expression, which is just a *single* covariance term, *does not depend on the index* (it is "summed out" in the analogous sense that we "integrate out" a variable). We can write

$$\gamma_k = \sigma_u^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+k} = \sigma_u^2 \theta_k + \sigma_u^2 \sum_{j=1}^{\infty} \frac{1}{\Gamma(d)\, j^{1-d}} \frac{1}{\Gamma(d)(j+k)^{1-d}} = \sigma_u^2 \theta_k + \frac{\sigma_u^2}{(\Gamma(d))^2} \sum_{j=1}^{\infty} \frac{1}{(j^2+jk)^{1-d}}$$

Since k is fixed, the leαding term in the denominator is equivalent to $\frac{1}{j^{2-2d}}$ which leads to convergence as long as $0 < d < 1/2$. So the autocovariance function exists for all k. Also, it does not depend on the index, and certainly not on t, but on k only.

So the y-process is weakly stationary.

But $\sum_{k=0}^{\infty} \gamma_k > \sigma_u^2 \theta_0 + \sigma_u^2 \theta_1 + \sigma_u^2 \theta_2 + \ldots = \sigma_u^2 \sum_{j=0}^{\infty} \theta_j \longrightarrow \infty$

So the infinite sum $\sum_{k=0}^{\infty} \gamma_k$ diverges. Therefore, the process

$$y_t = \sum_{j=0}^{\infty} \theta_j u_{t-j}, \qquad \theta_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \qquad 0 < d < \frac{1}{2},$$

is weakly stationary with "long memory".

Note: "Long memory" alone would not be a useful concept and tool. But together with weak stationarity, it enhances the concepts we have available to model and estimate real-world phenomena.

Exercises : Examine summability, weak stationarity etc in the following cases:

1. $y_t = \mu + \sum_{j=0}^{\infty} \theta_j u_{t-j}, \quad \mu \in \mathbb{R}$, everything else the same.

2. For $y_t = \sum_{j=0}^{\infty} \theta_j u_{t-j}$ suppose that $(u_t)_{t \in \mathbb{Z}}$ has the same properties as before, except that now $E(u_t) = \delta \neq 0$.--