## EGARCH(1,1)

Framework: $(z_t)_{t\in\mathbb{Z}}$ is iid with $E(z_0)=0$ and $E(z_0^2)=1$

Suppose that for the process $(h_t)_{t\in\mathbb{Z}}$ we have that $h_t \geq 0$ a.s., $E(h_t)<+\infty$ $\forall\, t\in\mathbb{Z}$. Consider the filtration $\mathcal{F}_t := \sigma(z_{t-1-i},\, i\geq 0)$ and suppose that $(h_t)_{t\in\mathbb{Z}}$ is adopted to $\mathcal{F}_t$.

(**)

Under the above framework, $(y_t)_{t\in\mathbb{Z}}$ with $y_t := z_t\sqrt{h_t}$ $\forall\, t\in\mathbb{Z}$ is called a conditionally (on $\mathcal{F}_t$) heteroskedastic process.

If $(h_t)_{t\in\mathbb{Z}}$ satisfies $h_t = \exp(\omega + a(|z_{t-1}| - E|z_{t-1}|) + \gamma z_{t-1} + b\ln h_{t-1})$, $t\in\mathbb{Z}$ then $(y_t)_{t\in\mathbb{Z}}$ is termed an EGARCH(1,1) process.

Due to the presence of the exponential no positivity constraints are required

So,

$$h_t = \exp\left[\omega + a\left(|z_{t-1}| - E|z_{t-1}|\right) + \gamma z_{t-1} + b\ln h_{t-1}\right], \quad t\in\mathbb{Z}$$

$$\Leftrightarrow \ln h_t = \omega + a\left(|z_{t-1}| - E|z_{t-1}|\right) + \gamma z_{t-1} + b\ln h_{t-1}, \quad t\in\mathbb{Z}$$

If we define $u_t := a\left(|z_{t-1}| - E|z_{t-1}|\right) + \gamma z_{t-1}$

$$\Rightarrow \ln h_t = \omega + b\ln h_{t-1} + u_t \qquad (*)$$

Since $(z_t)_{t\in\mathbb{Z}}$ is an iid process, $(u_t)_{t\in\mathbb{Z}}$ is also an iid process with $E(u_t)=0$ and $V(u_t)<+\infty$.

Thereby, $(*)$ is actually an AR(1) recursion with iid errors and it admits a unique stationary ergodic solution iff $|b|<1$ and consequently $(**)$ has a unique stationary ergodic solution where $(y_t)_{t\in\mathbb{Z}}$ is stationary ergodic.

Particularly, $(*)$ has a solution of the form:

$$h_t = \exp\left[\frac{\omega}{1-b} + \sum_{i=0}^{\infty} b^i\left[\left(a\left(|z_{t-1-i}| - E|z_{t-1-i}|\right)\right) + \gamma z_{t-i-1}\right]\right] \quad \forall\, t\in\mathbb{Z}$$

Since $z_t$ is iid $\sum_{i=0}^{\infty} b^i a E|z_{t-1-i}| = \sum_{i=0}^{\infty} b^i a E|z_0| = \dfrac{aE|z_0|}{1-b}$ $\longrightarrow$

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$$\Rightarrow h_t = \exp\left[\frac{\omega^*}{1-b} + \sum_{i=0}^{\infty} b^i \left[a|z_{t-1-i}| + \gamma z_{t-1-i}\right]\right] \quad \forall\, t \in \mathbb{Z}$$

where $\omega^* = \omega - aE|z_0|$.

$$\Rightarrow h_t = \exp\left(\frac{\omega^*}{1-b}\right) \prod_{i=0}^{\infty} \exp\left[b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right] \quad \forall\, t \in \mathbb{Z}$$

Let's consider the unconditional moments of $h_t$.

$$E(h_t) = \exp\left(\frac{\omega^*}{1-b}\right) \cdot E\left[\prod_{i=0}^{\infty} \exp\left[b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right]$$

$$\overset{z_t \text{ indep.}}{=} \exp\left(\frac{\omega^*}{1-b}\right) \prod_{i=0}^{\infty} E\left[\exp\left[b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right]$$

$$\overset{z_t \text{ ident.}}{=} \exp\left(\frac{\omega^*}{1-b}\right) \prod_{i=0}^{\infty} E\left[\exp\left[b^i\left(a|z_0| + \gamma z_0\right)\right]\right]$$

In order this expectation to exist we need to specify stricter restrictions on the properties of the distribution of $z$.

Consider that $(z_t) \sim iid\ N(0,1)$.
Then

$$E\left[\exp\left[b^i\left(a|z| + \gamma z\right)\right]\right] = E\left[\exp\left(ab^i|z| + \gamma b^i z\right)\right] \overset{\substack{\kappa_1 = ab^i \\ \kappa_2 = \gamma b^i}}{=}$$

$$E\left[\exp\left(\kappa_1|z| + \kappa_2 z\right)\right] \qquad (1)$$

Remember that, $E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

$$(1) = \int_{-\infty}^{\infty} \exp\left(\kappa_1|z| + \kappa_2 z\right) \varphi(z)\, dz = \int_{-\infty}^{\infty} \exp\left(\kappa_1|z| + \kappa_2 z\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\kappa_1|z| + \kappa_2 z - \frac{z^2}{2}\right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\kappa_1 z + \kappa_2 z - \frac{z^2}{2}\right) dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\left(\kappa_1 z + \kappa_2 z - \frac{z^2}{2}\right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left[-\frac{1}{2}\left(-2(\kappa_2-\kappa_1)z + z^2 \pm (\kappa_2-\kappa_1)^2\right)\right] dz$$

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$$+\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\left[-\frac{1}{2}\left(-2(\kappa_2+\kappa_1)z+z^2\pm(\kappa_2+\kappa_1)^2\right)\right]dz$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{(\kappa_2-\kappa_1)^2}{2}\right)\int_{-\infty}^{0}\exp\left[-\frac{1}{2}\left[z-(\kappa_2-\kappa_1)\right]^2\right]dz$$

$$+\frac{1}{\sqrt{2\pi}}\exp\left(\frac{(\kappa_2+\kappa_1)^2}{2}\right)\int_{0}^{\infty}\exp\left[-\frac{1}{2}\left[z-(\kappa_2+\kappa_1)\right]^2\right]dz$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{(\kappa_2-\kappa_1)^2}{2}\right)\int_{-\infty}^{-(\kappa_2-\kappa_1)}\exp\left(-\frac{1}{2}u^2\right)du$$

Set $u = z-(\kappa_2-\kappa_1)$
$du = dz$; $z\to 0,\ u\to-(\kappa_2-\kappa_1)$
$z\to-\infty,\ u\to-\infty$

$$+\frac{1}{\sqrt{2\pi}}\exp\left(\frac{(\kappa_2+\kappa_1)^2}{2}\right)\int_{(\kappa_2+\kappa_1)}^{-\infty}-\exp\left(-\frac{1}{2}(-v)^2\right)dv$$

Set $v=-z+(\kappa_2+\kappa_1)$
$dv=-dz$; $z\to 0,\ v\to(\kappa_2+\kappa_1)$
$z\to\infty,\ v\to-\infty.$

$$=\exp\left(\frac{(\kappa_2-\kappa_1)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-(\kappa_2-\kappa_1)}\exp\left(-\frac{1}{2}u^2\right)du+\exp\left(\frac{(\kappa_2+\kappa_1)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(\kappa_2+\kappa_1)}\exp\left(-\frac{1}{2}v^2\right)dv$$

$$=\exp\left(\frac{(\kappa_2-\kappa_1)^2}{2}\right)\Phi\left(-(\kappa_2-\kappa_1)\right)+\exp\left(\frac{(\kappa_2+\kappa_1)^2}{2}\right)\Phi(\kappa_2+\kappa_1)$$

$$=\exp\left(\frac{b^{2i}(\gamma-a)^2}{2}\right)\Phi\left(b^i(a-\gamma)\right)+\exp\left(\frac{b^{2i}(\gamma+a)^2}{2}\right)\Phi\left(b^i(\gamma+a)\right)<\infty$$

where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{t^2}{2}\right)dt$

Therefore,

$$E(h_t)=\exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}\left[\exp\left(\frac{b^{2i}(\gamma-a)^2}{2}\right)\Phi\left(b^i(a-\gamma)\right)+\exp\left(\frac{b^{2i}(\gamma+a)^2}{2}\right)\Phi\left(b^i(\gamma+a)\right)\right]$$

$$=\exp\left(\frac{\omega-aE|z_0|}{1-b}\right)\prod_{i=0}^{\infty}\left[\exp\left(\frac{b^{2i}(a-\gamma)^2}{2}\right)\Phi\left(b^i(a-\gamma)\right)+\exp\left(\frac{b^{2i}(a+\gamma)^2}{2}\right)\Phi\left(b^i(a+\gamma)\right)\right]$$

$$=\exp\left(\frac{\omega-a\sqrt{\frac{2}{\pi}}}{1-b}\right)\prod_{i=0}^{\infty}\left[\exp\left(\frac{A_i^2}{2}\right)\Phi(A_i)+\exp\left(\frac{B_i^2}{2}\right)\Phi(B_i)\right]$$

where $A_i=b^i(a-\gamma)$
$B_i=b^i(a+\gamma)$.

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Consider now the unconditional $\kappa$'th moment of $h_t$

$$h_t^{\kappa} = \left[\exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}\exp\left[b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right]^{\kappa}$$

$$= \exp\left(\frac{\kappa\omega^*}{1-b}\right)\prod_{i=0}^{\infty}\exp\left[\kappa b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]$$

$$\Rightarrow E(h_t^{\kappa}) = \exp\left(\frac{\kappa\omega^*}{1-b}\right) E\left[\prod_{i=0}^{\infty}\exp\left[\kappa b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right]$$

$$\overset{z_t \text{ indep.}}{=} \exp\left(\frac{\kappa\omega^*}{1-b}\right)\prod_{i=0}^{\infty} E\left[\exp\left[\kappa b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right]$$

$$\overset{z_t \text{ ident.}}{=} \exp\left(\frac{\kappa\omega^*}{1-b}\right)\prod_{i=0}^{\infty} E\left[\exp\left[\kappa b^i\left(a|z_0| + \gamma z_0\right)\right]\right]$$

Consider that $z_t \sim \text{iid } N(0,1)$

Then

$$E\left[\exp\left[\kappa b^i\left(a|z| + \gamma z\right)\right]\right] = E\left[\exp\left(\kappa a b^i|z| + \kappa\gamma b^i |z|\right)\right] \overset{\kappa_1 = \kappa a b^i}{\underset{\kappa_2 = \kappa\gamma b^i}{=}}$$

$$E\left[\exp\left(\kappa_1|z| + \kappa_2 z\right)\right] = \ldots\ldots =$$

$$\exp\left(\frac{(\kappa_2-\kappa_1)^2}{2}\right)\Phi\left(-(\kappa_2-\kappa_1)\right) + \exp\left(\frac{(\kappa_2+\kappa_1)^2}{2}\right)\Phi(\kappa_1+\kappa_2) =$$

$$\exp\left(\frac{b^{2i}\kappa^2(\gamma-a)^2}{2}\right)\Phi\left(\kappa b^i(a-\gamma)\right) + \exp\left(\frac{b^{2i}\kappa^2(\gamma+a)^2}{2}\right)\Phi\left(\kappa b^i(\gamma+a)\right)$$

Thereby,

$$E(h_t^{\kappa}) = \exp\left(\frac{\kappa\left(\omega - a\sqrt{\frac{2}{\pi}}\right)}{1-b}\right)\prod_{i=0}^{\infty}\left[\exp\left(\frac{A_i^2}{2}\right)\Phi(A_i) + \exp\left(\frac{B_i^2}{2}\right)\Phi(B_i)\right]$$

where $A_i = \kappa b^i(a-\gamma)$ $\quad\quad \forall \kappa \in \mathbb{R}$

$B_i = \kappa b^i(a+\gamma)$

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Consider the autocovariance of the process $(h_t)$ :

$$Cov(h_t, h_{t-k}) = E(h_t h_{t-k}) - E(h_t)E(h_{t-k})$$

$$E(h_t h_{t-k}) = E\left[\exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}\exp\left[b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right.$$

$$\left.\cdot\exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}\exp\left[b^i\left(a|z_{t-k-1-i}| + \gamma z_{t-k-1-i}\right)\right]\right]$$

$$= \exp\left(\frac{2\omega^*}{1-b}\right)E\left\{\exp\left[\sum_{i=0}^{\infty}b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right.$$

$$\left.\cdot\exp\left[\sum_{i=0}^{\infty}b^i\left(a|z_{t-k-1-i}| + \gamma z_{t-k-1-i}\right)\right]\right\}$$

$$= \exp\left(\frac{2\omega^*}{1-b}\right)E\left\{e^{b^0(a|z_{t-1}|+\gamma z_{t-1})}\, e^{b(a|z_{t-2}|+\gamma z_{t-2})}\cdots e^{b^{k-1}(a|z_{t-k}|+\gamma z_{t-k})}\right.$$

$$\cdot\, e^{b^k(a|z_{t-k-1}|+\gamma z_{t-k-1})}\, e^{b^{k+1}(a|z_{t-k-2}|+\gamma z_{t-k-2})}\cdots\cdots$$

$$\left. e^{b^0(a|z_{t-k-1}|+\gamma z_{t-k-1})}\, e^{b(a|z_{t-k-2}|+\gamma z_{t-k-2})}\cdots\cdots\right\}$$

$$\overset{\text{indep.}}{=}\exp\left(\frac{2\omega^*}{1-b}\right)E\left\{\exp\left[\sum_{i=0}^{k-1}b^i\left(a|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right]\right\}$$

$$\cdot\, E\left\{\exp\left[\sum_{i=0}^{\infty}\left(b^i + b^{i+k}\right)\left(a|z_{t-k-1-i}| + \gamma z_{t-k-1-i}\right)\right]\right\}$$

$$= \exp\left(\frac{2\omega^*}{1-b}\right)\prod_{i=0}^{k-1}\left[\exp\left(\frac{A_i^2}{2}\right)\Phi(A_i) + \exp\left(\frac{B_i^2}{2}\right)\Phi(B_i)\right]$$

$$\underbrace{E\left\{\prod_{i=0}^{\infty}\exp\left[\left(b^i + b^{i+k}\right)\left(a|z_{t-k-1-i}| + \gamma z_{t-k-1-i}\right)\right]\right\}}_{(**)}$$

$$(**) \overset{\text{ind.}}{\underset{\text{homog.}}{=}} \prod_{i=0}^{\infty}E\left[\exp\left[\left(b^i + b^{i+k}\right)\left(a|z_0| + \gamma z_0\right)\right]\right]$$ following the same steps as before setting $k_1 = a(b^i + b^{i+k})$, $k_2 = \gamma(b^i + b^{i+k})$

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$$= \prod_{i=0}^{\infty} \left[ \exp\left[ \frac{\left[b^i(1+b^K)(a-\gamma)\right]^2}{2} \right] \Phi\left(b^i(1+b^K)(a-\gamma)\right) + \exp\left[ \frac{\left[b^i(1+b^K)(a+\gamma)\right]^2}{2} \right] \Phi\left(b^i(1+b^K)(a+\gamma)\right) \right]$$

$$= \prod_{i=0}^{\infty} \left[ \exp\left(\frac{\Gamma_i^2}{2}\right) \Phi(\Gamma_i) + \exp\left(\frac{\Delta_i^2}{2}\right) \Phi(\Delta_i) \right]$$

Thereby,

$$E(h_t h_{t-K}) = \exp\left( \frac{2\left(\omega - a\sqrt{\frac{2}{\pi}}\right)}{1-b} \right) \prod_{i=0}^{K-1} \left[ \exp\left(\frac{A_i^2}{2}\right) \Phi(A_i) + \exp\left(\frac{B_i^2}{2}\right) \Phi(B_i) \right]$$

$$\cdot \prod_{i=0}^{\infty} \left[ \exp\left(\frac{\Gamma_i^2}{2}\right) \Phi(\Gamma_i) + \exp\left(\frac{\Delta_i^2}{2}\right) \Phi(\Delta_i) \right]$$

where $A_i = b^i(a-\gamma)$

$B_i = b^i(a+\gamma)$

$\Gamma_i = b^i(1+b^K)(a-\gamma)$

$\Delta_i = b^i(1+b^K)(a+\gamma)$

We have already computed $E(h_t)$, therefore the derivation of $Cov(h_t, h_{t-K})$ is trivial.