Example: GARCH(1,1) Process

In what follows we do not necessarily assume that $E(h_t) < +\infty$. However, given the convention employed in the general definition of conditionally heteroskedastic processes we will refer to $(h_t)_{t\in\mathbb{Z}}$ as the conditional variance process. In the framework described in this general definition we are examining the example of the GARCH(1,1) process.

Definition. $(y_t)_{t\in\mathbb{Z}}$ is called a GARCH(1,1) process iff for some $\omega, \alpha, b \in \mathbb{R}$ $(h_t)_{t\in\mathbb{Z}}$ satisfies the recursion $h_t = \omega + \alpha y_{t-1}^2 + b h_{t-1} = \omega + (\alpha z_{t-1}^2 + b) h_{t-1}$, $t \in \mathbb{Z}$. □

The previous recursion is termed GARCH(1,1) recursion, the parameter α ARCH parameter while b GARCH parameter. In order for a well-defined solution $(h_t)_{t\in\mathbb{Z}}$ to the GARCH recursion to exist several restrictions are to be forced on ω, α, b, and possibly on the distribution of z_0.

A. Positivity

It is easy to see that the GARCH recursion can be equivalently expressed as

$$h_t = \left(1, y_{t-1}, h_{t-1}^{1/2}\right) \begin{pmatrix} \omega & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} 1 \\ y_{t-1} \\ h_{t-1}^{1/2} \end{pmatrix}$$

which directly implies (why?) that a sufficient condition for $h_t > 0$, given that $h_{t-1} > 0$, $\forall t \in \mathbb{Z}$ is that the matrix · zeroing in the quadratic form above is positive definite, and this holds iff $\omega, \alpha, b > 0$ (why?). The previous is only sufficient since it is easy to see that if $\omega > 0$, $\alpha, b \geq 0$, $h_k > 0$ $\forall k < t$, then $\exists$ $0 \leq c < 1$ with $c \leq b$ such that $h_t = \omega + (\alpha z_{t-1}^2 + b) h_{t-1} \geq \omega + c h_{t-1} \geq \omega + c\omega + c^2 h_{t-2} \geq \dots \geq \omega \sum_{i=0}^{\infty} c^i = \frac{\omega}{1-c} > 0$. Since the latter lower bound is independent of t, we obtain that a weaker sufficient for positivity of $(h_t)_{t\in\mathbb{Z}}$ (i.e.

ensuring that $h_t > 0 \ \forall_{t \in \mathbb{Z}}$) is that $w > 0$, $\alpha, b \geq 0$. We employ the latter and we will further comment on the issue in the following.

B. Existence of a Unique (strictly) Stationary and Ergodic Solution of the GARCH recursion.

We have already provided (an incomplete version of) a general result that guarantees the existence of unique stationary and ergodic solutions to general stochastic recurrence equations that encompass the GARCH recursion above. We have not specified every detail concerning the sufficient conditions employed by the result, but if we examined them, it would be easily establishable if $E[\log^+(z_0^2)] < +\infty$, where $\log^+(x) = \begin{cases} \ln x & , x > 1 \\ 0 & , x \leq 1 \end{cases}$.

Notice that $0 < E[\log^+(z_0^2)] \leq E(z_0^2) = 1$ hence the condition above holds in our framework and thereby we do not need to worry about the "hidden" conditions. The condition that we have examined employed the expectation of the logarithm of the Lipschitz coefficient of the recursion. In the present case the recursion can be expressed as $h_t = f(z_{t-1}, h_{t-1})$ where $f(x,y) = w + (\alpha x^2 + b)y$ and since $(z_t)_{t \in \mathbb{Z}}$ is by assumption i.i.d. (hence strongly stationary and ergodic), we have that $\sup_{y>0} \left| \frac{\partial f(z_0, y)}{\partial y} \right| = \sup_{y>0} |\alpha z_0^2 + b| =$ $\alpha z_0^2 + b$ (why?). The condition is that $E\left[\ln \sup_{y>0} \left| \frac{\partial f(z_0, y)}{\partial y} \right| \right] < 0$ (and remember that it is allowed to assume the value $-\infty$) which is thereby specified as $E[\ln(\alpha z_0^2 + b)] < 0$ (*). Hence, due to the aforementioned general result, given the previous comment about the "hidden" conditions, (*) implies that the GARCH recursion admits a unique stationary and ergodic solution obtained as a "limit" of backward substitutions. Hence we add in our framework the following assumption.

Assumption - L. $E[\ln(\alpha z_0^2+b)]<0$. □

Remark. Obviously the condition in L restricts the parameter space for (α,b) given properties of the distribution of z_0. Notice that due to the inequality of Jensen, we have that $E[\ln(\alpha z_0^2+b)] \leq \ln(E(\alpha z_0^2+b)) = \ln(\alpha+b)$, hence it is possible that $1 \leq \alpha+b < k$, for some k depending on the distribution of z_0, and L holds. E.g. when $b=0$, $(*)$ is $\ln(\alpha)+2E(\ln|z_0|)<0 \iff \alpha<\exp[-2E(\ln|z_0|)]$ and if $z_0 \sim N(0,1)$ the rhs of the previous inequality becomes approximately 3.56. Obviously $\alpha+b<1 \Rightarrow E[\ln(\alpha z_0^2+b)]<0$. □

The next result provides the aforementioned solution.

Theorem. Under the aforementioned framework and if L holds then

$$\forall t\in\mathbb{Z},\ h_t = \omega\left[1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}(\alpha z_{t-j}^2+b)\right]\ P\text{ a.s.} \ \square$$

Proof. (Sketch)* The detailed version of the general result would imply that the unique stationary and ergodic solution is obtained, $\forall t\in\mathbb{Z}$ by

$$\begin{aligned} h_t &= \omega+(\alpha z_{t-1}^2+b)h_{t-1} = \omega(1+(\alpha z_{t-1}^2+b))+(\alpha z_{t-1}^2+b)(\alpha z_{t-2}^2+b)h_{t-2} \\ &= \omega(1+(\alpha z_{t-1}^2+b)+(\alpha z_{t-1}^2+b)(\alpha z_{t-2}^2+b))+(\alpha z_{t-1}^2+b)(\alpha z_{t-2}^2+b)(\alpha z_{t-3}^2+b)h_{t-3} \\ &= \lim_{n\to+\infty}\omega\left[1+\sum_{i=1}^{n-1}\prod_{j=1}^{i}(\alpha z_{t-j}^2+b)\right]+\lim_{n\to+\infty}\prod_{j=1}^{n}(\alpha z_{t-j}^2+b)\,c \end{aligned}$$

where c is an appropriate positive constant independent of n. Notice that

$$\prod_{j=1}^{n}(\alpha z_{t-j}^2+b) = \exp\left[\sum_{j=1}^{n}\ln(\alpha z_{t-j}^2+b)\right] = \exp\left[n\cdot\frac{1}{n}\sum_{j=1}^{n}\ln(\alpha z_{t-j}^2+b)\right].$$

* For simplicity assume that $-\infty < E[\ln(\alpha z_0^2+b)]<0$.

Due to Birkhoff's LLN (why is it applicable?) $\frac{1}{n}\sum_{j=1}^{n}\ln(\alpha z_{t-j}^2+b)$ converges P a.s. to $E[\ln(\alpha z_0^2+b)]$ as $n\to+\infty$, and the latter is due to (*) negative and this implies that $\lim_{n\to+\infty}\exp\left(n\frac{1}{n}\sum_{j=1}^{n}\ln(\alpha z_{t-j}^2+b)\right)=0$ P a.s. The result then follows. □

Comments:

1. The unique stationary and ergodic solution has the required general property of being obviously adapted to $(\mathcal{F}_t)_{t\in\mathbb{Z}}$. □

2. (Positivity Constraints Revisited) The solution implies that the afore-mentioned positivity constraints work. Notice that if we allow $\omega=0$ then we obtain that $h_t=0$ P a.s., $\forall t\in\mathbb{Z}$. If $\alpha=0$ ($\omega>0$), then obtain that $h_t=\omega(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}b)=\omega(1+\sum_{i=1}^{\infty}b^i)=\frac{\omega}{1-b}>0$, P a.s., $\forall t\in\mathbb{Z}$, hence we are in the realm of (non-degenerate) conditional homoskedasticity (why?). (remember that the solution is the unique stationary and ergodic one – this does not imply the non existence of other solutions without this property – can you find some?)

3. Obviously when $\alpha>0$ the solution is <u>not</u> a linear process. □

Corollary. If $\mathcal{L}$ holds then the GARCH(1,1) process $(y_t)_{t\in\mathbb{Z}}$ is strongly stationary and ergodic.

Proof. Remember the analogous result in the general definition. □

Comment. When $b=0$ we obtain the ARCH(1) process. □

Comment. It can be proven that $\mathcal{L}$ is also necessary for the Theorem to hold. □

C. Weak stationarity of the GARCH(1,1) Process.

Under the assumption framework established so far, we have that

$$E(h_t)=E\left[\omega\left[1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}(\alpha z_{t-j}^2+b)\right]\right]=\omega\left[1+\sum_{i=1}^{\infty}E\prod_{j=1}^{i}(\alpha z_{t-j}^2+b)\right]$$

$$\overset{\text{ind}}{=}\omega\left[1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}E(\alpha z_{t-j}^2+b)\right]=\omega\left[1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}(\alpha+b)\right]$$

$$=\omega\left[1+\sum_{i=1}^{\infty}(\alpha+b)^i\right]=\omega\sum_{i=0}^{\infty}(\alpha+b)^i=\begin{cases}+\infty, & \alpha+b\geq 1\\ \dfrac{\omega}{1-(\alpha+b)}, & \alpha+b<1\end{cases}.$$

This implies the following result.

Theorem. Under the aforementioned framework and if $\perp$ holds the GARCH(1,1) process $(y_t)_{t\in\mathbb{Z}}$ is weakly stationary iff $\alpha+b<1$. In this case it is actually a white noise process, with variance $\frac{\omega}{1-(\alpha+b)}$. □

Proof. From the previous $E(h_t)<+\infty \Leftrightarrow \alpha+b<1$. If this holds, and since $E(z_t^2)=1$, and z_t is independent of h_t, the conditional on $\mathcal{F}_t$ expectations of y_t and y_t^2 exist and L.I.E. is applicable. (Notice that due to Jensen's inequality $0\leq E(h_t^{1/2})\leq E(h_t)<+\infty$ and and due to the Cauchy-Schwarz inequality $E(h_t^{1/2}h_{t-k}^{1/2}z_{t-k})\leq E^{1/2}(h_t)E^{1/2}(h_{t-k}z_{t-k}^2)\overset{\text{ind}}{=}E^{1/2}(h_t)E^{1/2}(h_{t-k})E^{1/2}(z_{t-k}^2)<+\infty$). Hence

1. $E(y_t)=E(z_t h_t^{1/2})\overset{\text{L.I.E.}}{=}E\left[E(z_t h_t^{1/2})/\mathcal{F}_t\right]=E(h_t^{1/2}E(z_t/\mathcal{F}_t))\overset{\text{ind}}{=}E(h_t^{1/2}E(z_t))=0.$
2. $E(y_t^2)=E(z_t^2 h_t)\overset{\text{L.I.E.}}{=}E(E(z_t^2 h_t/\mathcal{F}_t))=E(h_t E(z_t^2/\mathcal{F}_t))\overset{\text{ind}}{=}E(h_t E(z_t^2))=$
   $=E(h_t)=\frac{\omega}{1-(\alpha+b)}.$
3. Due to 1 (why?) for $k>0$, $Cov(y_t,y_{t-k})=E(y_t y_{t-k})=E(z_t h_t^{1/2}h_{t-k}^{1/2}z_{t-k})$
   $\overset{\text{L.I.E.}}{=}E\left[E(z_t h_t^{1/2}h_{t-k}^{1/2}z_{t-k})/\mathcal{F}_t\right]=E(h_t^{1/2}h_{t-k}^{1/2}z_{t-k}E(z_t/\mathcal{F}_t))\overset{\text{ind}}{=}E(h_t^{1/2}h_{t-k}^{1/2}z_{t-k}E(z_t))$
   $=0.$ Hence the result is established. □

Comment. From the previous we have that $(y_t)_{t\in\mathbb{Z}}$ is strictly stationary and ergodic iff 1 holds, which allows for some parameter values such that $\alpha+b>1$. In such cases we have that the process is not weakly stationary, hence obtaining non trivial examples of processes that are strictly but not weakly stationary. $\circ$

D. ARMA Representation – Autocovariance of the Squared Process.

It is a stylized fact of the empirical characteristics of categories of financial returns (observed even in "moderate", e.g. weekly, frequencies) to have statistically insignificant empirical autocovariances (even for small values of k), something that does not hold for the empirical autocovariances of the squared returns which appear to have large and statistically significant values, even for "not so small k". Those stylized facts cannot be reproduced by ARMA models (why?). The class of conditionally heteroskedastic models was, among others, initially examined for the study of models that can (possibly partially) cope with suchlike stylized facts. For the GARCH(1,1) case we have already established the white noise behavior when $\alpha+b<1$. Under further restrictions on the properties of the distribution of z_0 and the parameter space we will obtain an "ARMA-type" behavior for $(y_t^2)_{t\in\mathbb{Z}}$.

Suppose that $E(z_0^4):=u<+\infty$. We have that $E(h_t^2)=$

$$= E\left(\omega^2 + (\alpha z_{t-1}^2+b)^2 h_{t-1}^2 + 2\omega(\alpha z_{t-1}^2+b)h_{t-1}\right) =$$

$$= \omega^2 + E\left[(\alpha^2 z_{t-1}^4 + 2\alpha b z_{t-1}^2 + b^2)h_{t-1}^2\right] + 2\omega E\left[(\alpha z_{t-1}^2+b)h_{t-1}\right]$$

$$\overset{\text{ind}}{=} \omega^2 + (\alpha^2 u + 2\alpha b + b^2)E(h_{t-1}^2) + 2\omega(\alpha+b)E(h_{t-1})$$

$$= \omega^2 + A(\alpha,b,u)E(h_{t-1}^2) + \frac{2\omega(\alpha+b)}{1-(\alpha+b)} \qquad (A(\alpha,b,u):=\alpha^2 u+2\alpha b+b^2)$$

$$= P(\omega,\alpha,b) + A(\alpha,b,u)E(h_{t-1}^2) \qquad \left(P(\omega,\alpha,b):=\omega^2+\frac{2\omega(\alpha+b)}{1-(\alpha+b)}\right)$$

Denoting $\mu_t := E(h_t^2)$ we have the recursion

$$\mu_t = P(\omega,\alpha,b) + A(\alpha,b,u)\,\mu_{t-1}$$
$$\Leftrightarrow \mu_t(1 - A(\alpha,b,u)L) = P(\omega,\alpha,b).$$

Employing the UDC ($\Leftrightarrow A(\alpha,b,u)<1$ - why?, notice that $A(\alpha,b,u)\geq 0$ - why?) we have that if it holds, then

$$\mu_t = \frac{P(\omega,\alpha,b)}{1-A(\alpha,b,u)}.$$ This leads to the following

result.

Lemma. If $E(z_0^4)<+\infty$ and $A(\alpha,b,u)<1$ then the process $(v_t)_{t\in\mathbb{Z}}$, defined by $v_t := (z_t^2-1)h_t$ is a stationary and ergodic s.m.d. adapted to $(G_t)_{t\in\mathbb{Z}}$, with $E(v_0^2) = (u-1)\dfrac{P(\omega,\alpha,b)}{1-A(\alpha,b,u)}$. $\square$

Comment. Notice that since $u\geq 1$ (why?) $A(\alpha,b,u)<1 \Rightarrow \alpha+b<1 \Rightarrow E[\ln(\alpha z_0^2+b)]<0$. E.g. when $b=0$, $A(\alpha,b,u)<1 \Leftrightarrow \alpha<\sqrt{\frac{1}{u}}$. $\square$

Proof. Stationarity and ergodicity for $(v_t)_{t\in\mathbb{Z}}$ follows from the previous comment (why?). Adaptation to $(G_t)_{t\in\mathbb{Z}}$ follows from the adaptation of $(h_t)_{t\in\mathbb{Z}}$ to $(F_t)_{t\in\mathbb{Z}}$ and the form of v_t. $E(v_t^2) =$
$= E((z_t^2-1)^2h_t^2) \overset{ind}{=} E(z_t^2-1)^2E(h_t^2) = E(z_t^4-2z_t^2+1)E(h_t^2) =$
$= (u-1)\dfrac{P(\omega,\alpha,b)}{1-A(\alpha,b,u)} < +\infty$. Finally $E[(z_t^2-1)h_t / G_{t-1}] = E((z_t^2-1)h_t / F_t)$

$= h_t E((z_t^2-1)/F_t) \overset{ind}{=} h_t E(z_t^2-1) = 0.$ $\square$

Hence we obtain the following "ARMA-type" representation for the squared process.

Theorem. If $E(z_0^4)<+\infty$ and $A(\alpha,b,u)<1$ then $(y_t^2)_{t\in\mathbb{Z}}$ is the linear process solution to the ARMA(1,1)* recursion

* with a non-zero constant.

$$y_t^2 = \omega + (\alpha+b)y_{t-1}^2 - b v_{t-1} + v_t \quad [**]$$

where $(v_t)_{t\in\mathbb{Z}}$ as above. $\square$

Proof. $h_t = \omega + \alpha y_{t-1}^2 + b h_{t-1} \Leftrightarrow y_t^2 - h_t = y_t^2 - \omega - \alpha y_{t-1}^2 - b h_{t-1} \Leftrightarrow$

$(z_t^2 - 1)h_t = y_t^2 - \alpha y_{t-1}^2 - b h_{t-1} - \omega \Leftrightarrow y_t^2 = v_t + \omega + \alpha y_{t-1}^2 + b h_{t-1}$

$\Leftrightarrow y_t^2 = \omega + (\alpha+b)y_{t-1}^2 - b(y_{t-1}^2 - h_{t-1}) + v_t \Leftrightarrow y_t^2 = \omega + (\alpha+b)y_{t-1}^2 -$
$- b((z_{t-1}^2 - 1)h_{t-1}) + v_t \Leftrightarrow y_t^2 = \omega + (\alpha+b)y_{t-1}^2 - b v_{t-1} + v_t.$ $\square$

Comments.

1. When $\alpha = 0$ we have the common root property (which implies what?)
2. The autocovariance function of $(y_t^2)_{t\in\mathbb{Z}}$ satisfies (why?) the formulae that we have elsewhere derive under the identifications $\beta_1 = \alpha+b$, $\theta_1 = b$, $\sigma^2 = (\mu-1)\dfrac{P(\omega,\alpha,b)}{1-A(\alpha,b,\mu)}$ (Derive it!).

E.g. in the ARCH(1) case (i.e. $b=0$), we have that $y_t^2 = \omega + \alpha y_{t-1}^2 + v_t$ i.e. we obtain an AR(1) process for the squares, whence,

$$\gamma_k^* := \mathrm{Cov}(y_t^2, y_{t-k}^2) = \frac{\beta_1^k}{1-\beta_1^2}\sigma^2 = \frac{\alpha^k}{1-\alpha^2}\,\frac{P(\omega,\alpha,0)}{1-A(\alpha,0,\mu)}(\mu-1)$$

$$= (\mu-1)\frac{\alpha^k}{1-\alpha^2}\,\frac{\omega^2 + \frac{2\omega\alpha}{1-\alpha}}{1-\alpha^2\mu} = \frac{\alpha^k((1-\alpha)\omega^2 + 2\omega\alpha)}{(1-\alpha)^2(1+\alpha)(1-\alpha^2\mu)}(\mu-1).$$

3. Hence under the aforementioned restrictions the GARCH(1,1) process is (partially) consistent with aforementioned stylized facts. $\square$

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please r eport any typos to stelios@aueb.gr or the course's e-class.]