0.0.1 Circular city model

Salop (Bell, 1979).

In stage 1 firms choose whether to enter the market. Each firm that enters pays a fixed cost F. For simplicity, we assume that the n firms that have entered locate equidistantly around the circle. In stage 2 firms compete in prices. We search for a symmetric equilibrium. Suppose each of the $n-1$ firms charges p' and firm i charges p. The marginal consumer satisfies,

$$p + tx = p' + t\left(\frac{1}{n} - x\right).$$

The demand of firm i is,

$$2x = \frac{p' - p + t/n}{t}.$$

The profit function of firm i is,

$$\pi = \frac{p\left(p' - p + t/n\right)}{t}.$$

After taking the first order conditions and imposing symmetry, $p = p'$, we derive the equilibrium prices,

$$p = \frac{t}{n}.$$

The firm profit is,

$$\pi = \frac{t}{n^2} - F.$$

In the free entry equilibrium the number of firms is,

$$n^* = \sqrt{\frac{t}{F}}.$$

Social planner's problem The social planner chooses the number of firms to minimize the sum of the total transportation cost and the cost of entry, given by,

$$C\left(n\right) = \frac{t}{4n} + nF.$$

The socially optimal number of firms is,

$$n^o = \frac{1}{2}\sqrt{\frac{t}{F}} < n^*.$$

Too much variety from the social perspective.

0.1 Vertical differentiation

Overview: *Persistence of profits with free entry. Importance of name brand. Natural oligopoly.*

Shaked and Sutton (ReStud, 1982).

There are n firms in the market each producing one good of quality q_i, with $0 < m < q < M < \infty$. Let's assume that the qualities are exogenously fixed as follows,

$$q_1 < q_2 < \cdots < q_n.$$

There is a continuum of consumers with income t that is uniformly distributed in the (a, b) interval, with $2a < b < 4a$. The utility of the consumer with income t is,

$$U(q, p; t) = tq - p,$$

where p denotes price. A consumer is indifferent between two successive products if and only if,

$$tq - p = tq' - p'.$$

Consumers with

$$t > \frac{p' - p}{q' - q}$$

will choose q'.

Figure **??** depicts the market shares of each firm. The profit function of firm n is,

$$\pi_n(p_n, p_{n-1}) = p_n(b - t_{n-1}) = p_n\left(b - \frac{p_n - p_{n-1}}{q_n - q_{n-1}}\right).$$

The profit function of firm $n-1$ is,

$$\pi_{n-1}(p_n, p_{n-1}, p_{n-2}) = p_{n-1}\left(\frac{p_n - p_{n-1}}{q_n - q_{n-1}} - \frac{p_{n-1} - p_{n-2}}{q_{n-1} - q_{n-2}}\right).$$

Only the top two quality firms will make positive profits.

Suppose not. So there are at least two interior points (3 firms). The first order condition for firm n is,

$$\begin{aligned} 0 &= b - t_{n-1} - \frac{p_n - p_{n-1} + p_{n-1}}{q_n - q_{n-1}} \\ &= b - 2t_{n-1} - \frac{p_{n-1}}{q_n - q_{n-1}} \leq b - 2t_{n-1}. \end{aligned}$$

So, $b \geq 2t_{n-1}$. The first order condition of firm $n-1$ yields $t_{n-1} \geq 2t_{n-2}$. Putting the two inequalities together we have,

$$b \geq 4t_{n-2} > 4a,$$

because $t_{n-2} > a$, which is a contradiction to the assumption that $b < 4a$. Thus, only the top two firms will make positive profits.

When firms are choosing qualities there will be a "race to the top" result. The lowest quality firms (with three firms or more) make zero profits. Hence, only two firms will enter "finiteness property." Therefore, free entry equilibrium does not dissipate profits.

Now allow the support of the distribution to change.

- If $a < b < 2a$, then 1 firm enters.
- If $2a < b < 4a$, then 2 firms enter.
- If $4a < b < 8a$, then 3 firms enter.

With fixed support of the income distribution only a finite number of firms will enter and each firm makes positive profits.

Now let's assume that $n = 2$ and firms choose qualities. Let $\Delta \equiv q_1 - q_2$. The consumer who is indifferent between firm 1 and firm 2 is given by,

$$t_1 = \frac{p_1 - p_2}{\Delta}$$

and the consumer who is indifferent between firm 2 and no purchase is given by,

$$t_0 = \frac{p_2}{q_2}.$$

The demand of firm 1 is,

$$d_1 = b - t_1 = b - \frac{p_1 - p_2}{\Delta}.$$

The demand of firm 2 is,

$$d_2 = \begin{cases} t_1 - t_0 = \frac{q_2 p_1 - p_2 q_1}{\Delta q_2}, \text{ if } t_0 > a \Rightarrow p_2 \geq a q_2 \text{ (uncovered market)} \\ t_1 - a = \frac{p_1 - p_2}{\Delta} - a, \text{ if } t_0 \leq a \Rightarrow p_2 \leq a q_2 \text{ (covered market).} \end{cases}$$

The profit functions are,

$$\pi_1 = p_1 \left(b - \frac{p_1 - p_2}{\Delta} \right)$$

and

$$\pi_2 = \begin{cases} p_2\left(\frac{q_2p_1 - p_2q_1}{\Delta q_2}\right), \text{ if } p_2 > aq_2 \\ p_2\left(\frac{p_1-p_2}{\Delta} - a\right), \text{if } p_2 \leq aq_2. \end{cases}$$

We will look for an equilibrium that is consistent with a covered market. Taking the first order conditions and solving with respect to prices we obtain,

$$p_1 = \frac{(2b-a)(q_1-q_2)}{3} \text{ and } p_2 = \frac{(b-2a)(q_1-q_2)}{3}.$$

These solutions are valid only if $t_0 \leq a \Rightarrow \frac{(b-2a)(q_1-q_2)}{3} \leq aq_2$. (Second order conditions are also satisfied). The equilibrium profits are given by,

$$\pi_1 = \frac{(2b-a)^2(q_1-q_2)}{9(b-a)} \text{ and } \pi_2 = \frac{(b-2a)^2(q_1-q_2)}{9(b-a)}.$$

When qualities are endogenized, firm 1 will choose the highest possible level of quality and firm 2 will choose $q_2 = 0$ (maximal differentiation). The low quality firm by lowering its quality all the way down to the lowest level is able to credibly commit to its rival that it will not be a fierce competitor. The response of the rival (firm 1) is to raise its price, which in turn allows firm 2 to also raise its price. Price competition is mitigated.

0.2 Monopolistic competition

Overview: *Demand considerations as a source of monopoly pricing. Price (but not profits) stays bounded away from marginal cost, even with free entry. Modeling consumers' discrete choice.*

Dixit and Stiglitz (AER, 1977).

In the Cournot model, the price falls gradually with entry. In the Shaked and Sutton model price drops dramatically with entry (finiteness property). In models of monopolistic competition, price remains constant with entry.

The number of firms is denoted by n. The representative consumer consumes $n+1$ goods $q_0, q_1, ..., q_n$. Good 0 is the numeraire and its price p_0 is 1. The prices of the other goods are denoted by $p_1, p_2, ..., p_n$. The budget constraint is $q_0 + \sum_{i=1}^n p_i q_i = I \Rightarrow q_0 = I - \sum_{i=1}^n p_i q_i$. The utility is of the CES form and it is given by,

$$U = U\left(q_0, \left(\sum_{i=1}^n q_i^\rho\right)^{1/\rho}\right) = U\left(I - \sum_{i=1}^n p_i q_i, \left(\sum_{i=1}^n q_i^\rho\right)^{1/\rho}\right),$$

where $\rho \leq 1$. The first order conditions are,

$$\frac{\partial U}{\partial q_i} = -U_1 p_i + U_2 \frac{1}{\rho} \left(\sum_{i=1}^{n} q_i^{\rho} \right)^{1/\rho - 1} \rho q_i^{\rho-1} = 0 \Rightarrow$$

$$q_i = \left[\frac{U_1}{U_2} \frac{1}{\left(\sum_{i=1}^{n} q_i^{\rho} \right)^{1/\rho - 1}} \right]^{\frac{1}{\rho-1}} p_i^{\frac{1}{\rho-1}} = k p_i^{\frac{1}{\rho-1}}.$$

The profit function of firm i is,

$$\pi_i(p_i) = (p_i - c)\, k p_i^{\frac{1}{\rho-1}}.$$

The first order condition is,

$$\frac{\partial \pi_i}{\partial p_i} = p_i^{\frac{1}{\rho-1}} + \frac{1}{\rho - 1} p_i^{\frac{1}{\rho-1} - 1} (p_i - c) = 0.$$

The solution is,

$$p_i = \frac{c}{\rho}.$$

Note that the price is independent of the number of varieties and is bounded away from marginal cost c. So, competition does not become more intense as more firms enter.

Example: Credit cards. Many offers, where each card charges an exorbitant interest, but each sells very few units.

Next, we can determine the number of firms that will enter the market. Given a fixed cost of entry F, the zero profit condition is,

$$\left(\frac{c}{\rho} - c \right) q - F = 0 \Rightarrow q = \frac{F}{\left(\frac{c}{\rho} - c \right)}.$$

Using the first order condition from above we have,

$$U_1 \frac{c}{\rho} = U_2 q^{\rho-1} (n q^{\rho})^{1/\rho - 1}.$$

This equation together with q from above determines the number of firms.

Perloff and Salop (ReStud, 1985).

Tries to synthesize location models with models of monopolistic competition. Consumer has a vector of valuations for the n products,

$$(V_1, ..., V_n).$$

All consumers are the same. Valuation V_j is distributed according to $F(V_j)$. Each consumer purchases the brand among those available that maximizes his net surplus, i.e.,

$$\max_{j=1,\dots,n} \{V_j - p_j\}.$$

The demand function of good i is,

$$Q_i(p_1, ..., p_n) = \Pr ob\,(V_i - p_i > V_j - p_j),\ \forall j \neq i = \int \Pi_{j\neq i} F(V_i + p_j - p_i)\, f(V_i)\, dV_i.$$

The profit function of firm i is,

$$\pi_i = (p_i - c)\, Q_i(p_1, ..., p_n).$$

The first order condition is,

$$\frac{\partial \pi_i}{\partial p_i} = Q_i + (p_i - c)\, Q_i' = 0 \Rightarrow p = c - \frac{Q_i}{Q_i'} \text{ (mark-up formula)}.$$

In a symmetric equilibrium $p_1^* = \cdots = p_n^*$. Then,

$$Q_i = \int F(V)^{n-1} f(V)\, dV = \left.\frac{F(V)^n}{n}\right|_0^1 = \frac{1}{n}.$$

Also,

$$-Q_i' = \sum \int \Pi_{j\neq i} F(V + p^* - p_i)\, f(V_i)\, f(V + p^* - p_i)\, dV_i \Rightarrow \text{ (setting } p_i = p^*)$$
$$-Q_i' = (n-1) \int F(V)^{n-2} f(V)^2\, dV.$$

So,

$$p^* = c + \frac{1}{n(n-1)\int F(V)^{n-2} f(V)^2\, dV}.$$

Prices may or may not converge to marginal cost. If the support of the valuations is bounded, then $p \to c$, as $n \to \infty$. Also, if the support is unbounded, then, under certain conditions, $p \to c$ and under other conditions p does not converge to c as $n \to \infty$. For example, when the distribution is exponential, $f(v) = \lambda e^{-\lambda v}$ on $[0, \infty)$ with $\lambda > 0$, then $\lim_{n\to\infty} p^* = c + 1/\lambda > c$.

When the distribution of the valuations is uniform on $[0, 1]$,

$$n(n-1) \int F(V)^{n-2} f(V)^2\, dV = n$$

and $p \to c$ as $n \to \infty$.

0.3 Bertrand and Edgeworth models

Overview: *Capacity constraints attenuate competition, enabling firms to make positive profits. How mixed strategy equilibria work. Profits may increase as firms enter (consumer loyalty).*

Each consumer buys only one unit if the price is below 1. There are two firms in the market who compete in prices and with capacity limit X (equal). The profit function of firm 1 is,

$$\pi_1(p_1, p_2) = \begin{cases} p_1 \min\{X, 1\}, & \text{if } p_1 < p_2 \\ p_1 \min\left\{X, \frac{1}{2}\right\}, & \text{if } p_1 = p_2 \\ p_1 \min\{X, \max\{1 - X, 0\}\}, & \text{if } p_1 > p_2. \end{cases}$$

There are three cases:

- $X \geq 1$. Capacity limit is not binding. The equilibrium is Bertrand, i.e., $p_1 = p_2 = 0$. The equilibrium profits are zero.
- $X \leq \frac{1}{2}$. This is similar to the case we examined above. Firms are too constrained by their capacities. The equilibrium is $p_1 = p_2 = 1$. The equilibrium profits are $\pi_1^* = \pi_2^* = X$.
- $\frac{1}{2} < X < 1$. Firm 1's profit function is,

$$\pi_1(p_1, p_2) = \begin{cases} p_1 X, \text{ if } p_1 < p_2 \\ p_1 \frac{1}{2}, \text{ if } p_1 = p_2 \\ p_1 (1 - X), \text{ if } p_1 > p_2. \end{cases}$$

Let's analyze the last case. First, note that there does not exist an equilibrium in pure strategies. If $p_1 = p_2 = 0$, then one firm has an incentive to increase its price. It will serve the residual demand, which is strictly positive, and it will become better off. If $p_1 = p_2 > 0$, then one firm will undercut the price of the rival by a very small amount and become better off. If $p_1 > p_2$, then firm 2 can increase its price slightly without loosing any customers.

Next, we look for a mixed strategy equilibrium. The support of the distribution is $\left[\underline{p}, 1\right]$. We look for a symmetric equilibrium. Assume firm 1 charges p and firm 2 mixes according to $F(\cdot)$. The (expected) profit function of firm 1 is ,

$$\pi_1(p, F) = \underbrace{pX[1 - F(p)]}_{\text{if } p \text{ is less than 2's price}} + \underbrace{p(1 - X) F(p)}_{\text{if } p \text{ is higher than 2's price}}.$$

The probability that $p_1 = p_2$ is zero, i.e., no atoms in the price distribution.[1] By way of contradiction suppose there are atoms in the

[1] An atom is a point in the support of the distribution that has a strictly positive probability.

distribution. There exists a $p \in [\underline{p}, 1]$ such that $p_1 = p_2 = p$ occurs with strictly positive probability. Then firm 1 would become strictly better off by lowering its price slightly. This is a profitable deviation because the probability of this point is strictly positive. Hence, this p cannot be part of an equilibrium.

For F to be an equilibrium it must be that firm 1 is indifferent between any p, i.e., any p gives firm 1 the same profits. Therefore, $\pi_1(p, F) =$constant. Moreover, this constant must be equal to the profit the firm can get for sure. The firm can always set its price equal to 1 and serve the residual demand. This implies,

$$\pi_1(p, F) = pX[1 - F(p)] + p(1 - X)F(p) = 1 - X.$$

Solving with respect to F we obtain the mixed strategy Nash equilibrium,

$$F(p) = \frac{pX - 1 + X}{p(2X - 1)}.$$

For $F(p)$ to be a distribution function it must be that $F(\underline{p}) = 0 \Rightarrow \underline{p} = \frac{1-X}{X}$. The density function is,

$$f(p) = \frac{dF(p)}{dp} = \frac{1 - X}{2X - 1}\frac{1}{p^2}.$$

The density is decreasing in p which implies that lower prices are more likely than high prices. Using the density we can compute the expected price,

$$Ep = \int_{\underline{p}}^{1} pf(p)\,dp = -\left(\frac{1 - X}{2X - 1}\right)\log\left(\frac{1 - X}{X}\right).$$

Note that $Ep > 0$ and i) $\lim_{X \to 1} Ep = 0$, ii) $\lim_{X \to 1/2} Ep = 1$. The equilibrium expected profits, by construction, are $\pi_1^* = \pi_2^* = 1 - X$.

0.3.1 A model where entry leads to higher prices

Rosenthal (Econometrica, 1980).

There are n sellers in a market, each producing at a zero cost. Consumers are split in two markets: a monopoly market (M) and a common market (C). Each consumer's demand is rectangular (i.e., buys one unit if the price is weakly below 1 and does not buy if the price is above 1). Each consumer in the monopoly market buys from only one firm (his favorite firm) if the price is below 1 and does not buy at all if the price is above 1. (Consumers in the M market are called loyal consumers and do not compare prices). There is only one loyal consumer per each firm.

There is also one consumer in the common market who does not have a favorite firm and buys from the one with the lowest price.

Firms compete in prices (simultaneously). Each firm charges only one price.

The profit function of firm 1 can be expressed as follows (the other firms' profit functions are similar),

$$\pi_1 (p_1, ..., p_n) = \begin{cases} p_1 + \frac{p_1}{m}, \text{ if } p_1 = \min \{p_1, ..., p_n\} \text{ and } m \text{ prices achieve the min.} \\ p_1, \text{ if } p_1 > \min \{p_2, ..., p_n\}. \end{cases}$$

A pure strategy equilibrium does not exist. If all prices are equal, then one firm has an incentive to lower its price slightly and sell to the person who compares prices (the switcher). If prices are not equal, then the firm with the lowest price (for example) can increase its price and become better off.

Let's now find the symmetric mixed strategy equilibrium. (The equilibrium distribution F contains no atoms; same argument as above).

We need the following two probabilities first. The probability that firm 1's price is the lowest is: $\Pr ob (p_1 = \min \{p_1, ..., p_n\}) = [1 - F(p_1)]^{n-1}$. In this case firm 1 sells 2 units (one to its loyal consumer and 1 to the switcher). The probability that firm 1's price is not the lowest is $\Pr ob (p_1 > \min \{p_2, ..., p_n\}) = 1 - [1 - F(p_1)]^{n-1}$. In this case firm 1 sells only 1 unit.

Assume firm 1 charges p and the $n-1$ firms mix according to $F(\cdot)$. The (expected) profit function of firm 1 is ,

$$\pi_1 (p_1, F, ..., F) = p_1 \left\{ 2 [1 - F(p_1)]^{n-1} + 1 - [1 - F(p_1)]^{n-1} \right\} = 1$$

because firm 1 can always charge $p = 1$ and sell to its loyal customer. Solving for F we can derive the equilibrium distribution,

$$F(p|n) = 1 - \left(\frac{1-p}{p} \right)^{1/(n-1)}.$$

The support of the distribution is $\left[\frac{1}{2}, 1\right]$. Note that $F\left(\frac{1}{2}\right) = 0$ and $F(1) = 1$. This is independent of the number of firms n (as long as $n \geq 2$).

By differentiating $F(p|n)$ w.r.t. n we obtain,

$$\frac{dF}{dn} = \frac{\left(\frac{1-p}{p}\right)^{1/(n-1)} \ln \left(\frac{1-p}{p}\right)}{(n-1)^2} \leq 0, \text{ since } p \in \left[\frac{1}{2}, 1\right].$$

Therefore as n increases $F(p)$ decreases for all p, which is equivalent to saying that a price distribution with a higher n first order stochastically dominates a one with a lower n. This also implies that higher n leads to a higher expected price.

Intuition: Recall that firms lower their prices in order to attract the switcher. If the number of firms in the market increases and the size of the common market (switcher) stays the same, then the common market, relatively speaking, shrinks. Then firms focus more on their loyal customers and raise their prices.