

# 1 Intertemporal Choices

We now extend our framework to incorporate “intertemporal decisions”, i.e. decisions that involve a trade-off between the present and the future. The vast majority of important economic decisions are of this type:

- the consumption decision involves a trade-off between consuming more today or saving more today and consuming more in the future;
- the education decision involves a trade-off between earning today or foregoing current earnings to go to school (and pay tuition) in order to earn more in the future;
- the fiscal policy decision of governments involves a trade-off between taxing consumers today or raising debt and taxing more consumers in the future (as interests on debt cumulate over time).

To incorporate the time dimension into economic decisions, we extend the time-horizon of the economy to 2 periods. The 2-period economy is the simplest framework for understanding intertemporal choices and dynamic issues. The generalization to multiple periods is straightforward.

We start from a simple “endowment” economy where there is no production and no labor supply, but households receive some endowment of income every period. It’s like life in Monte Carlo... where nobody works, everybody lives off income coming from their wealth. Given the absence of production, we just need to worry about the household side of the economy.

## 1.1 Household Problem and the Consumption-Saving Decision

Households live for 2 periods, and receive endowments of income  $(y_1, y_2)$  respectively in the first and second period. They decide optimally how much to consume  $(c_1)$ , and how much to save  $(s)$  in the first period in order to consume  $(c_2)$  in the second and final period of life. As we will see later, you can think of consumption at period 1 and consumption at period 2 as two different consumption goods with a relative price between them. Households can freely borrow (e.g., run a credit card balance) and lend (e.g., deposit in a saving account) in financial markets at the prevailing interest rate  $r$ . If they save  $s > 0$  and  $c_1 < y_1$ . If they borrow,  $s < 0$  and  $c_1 > y_1$ .

Household preferences are given by:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

where  $0 < \beta < 1$  is the **discount factor**, i.e. the factor at which future consumption is discounted compared to current consumption. The assumption  $\beta < 1$  means that households always prefer to consume  $c$  this period than exactly the same amount next period.

Preferences  $U(c_1, c_2)$  have the same properties discussed in the one period model for  $U(c, l)$ : monotonicity (more is preferred to less), concavity (decreasing marginal utility in consumption in both periods), and the assumption that  $c_t$  at every  $t$  is a normal good. Recall that concavity implies love of diversity. In this context, it means that the individual has a preference to smooth consumption across time. We'll return on consumption smoothing later. This is an essential property of optimal consumption behavior. Finally, recall that the normal good property (both  $c_1$  and  $c_2$  rise when income rises) applies for parallel shifts of the budget constraints that induce a pure income effect. In an endowment economy without work and production, a change in income  $y_t$  induces a pure income effect.

The maximization problem of the household (HP) is:

$$\begin{aligned} & \max_{\{c_1, c_2, s\}} U(c_1, c_2) & \text{(HP)} \\ & \text{s.t.} \\ & c_1 + s \leq y_1 \\ & c_2 \leq (1+r)s + y_2 \end{aligned}$$

where the two constraints represent the budget constraints of the households at time 1 and at time 2. First, it is useful to simplify this problem by substituting out savings  $s$  using the period-one budget constraint into the period-two budget constraint:

$$\begin{aligned} c_2 & \leq (1+r)(y_1 - c_1) + y_2 \\ c_1 + \frac{c_2}{1+r} & \leq y_1 + \frac{y_2}{1+r} \end{aligned} \tag{1}$$

We have obtained the so-called **lifetime budget constraint** which states that the discounted present value (DPV) of consumption must equal the DPV of income. Notice that even though we have not imposed any exogenous borrowing limit, there is a so-called natural borrowing constraints that arises exactly, “naturally”. Suppose the household in period 1 borrows more than  $y_2/(1+r)$ . Then, in period 2 his consumption would have to be negative and his utility would be  $-\infty$ . So, the household would never choose to borrow more than that amount.

The simplified problem can be re-written now as

$$\begin{aligned} & \max_{\{c_1, c_2\}} U(c_1, c_2) \\ & \text{s.t.} \\ & c_1 + \frac{c_2}{1+r} \leq y_1 + \frac{y_2}{1+r} \end{aligned}$$

which requires choosing how to allocate the DPV of income (total resources available to the household) between consumption at time 1 and at time 2. The trade-off is immediate: consuming earlier is better because future consumption gives less utility ( $\beta < 1$ ), but postponing consumption through savings allows higher consumption next period, since savings are rewarded at rate  $r$ .

The constrained maximization problem above is like any other static problem that can therefore be solved with the usual Lagrangian techniques:

$$L(c_1, c_2, \lambda) = u(c_1) + \beta u(c_2) + \lambda \left[ y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

The solution of this problem is

$$\begin{aligned} FOC(c_1) &\Rightarrow u'(c_1) = \lambda \\ FOC(c_2) &\Rightarrow \beta u'(c_2) = \frac{\lambda}{1+r} \\ FOC(\lambda) &\Rightarrow c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \end{aligned}$$

Putting together the first two conditions, we arrive at the key equation:

$$\frac{u'(c_1)}{\beta u'(c_2)} = (1+r) \quad (2)$$

This equation is called the **Euler Equation** and describes the intertemporal optimal consumption choice of the household between the current and the future period. The LHS is the MRS between consumption today and consumption next period. The RHS is the gross interest rate, i.e., the relative price between consumption today and consumption next period (see below). If  $r$  is high, the price of consuming today is high because the agent is foregoing a high rate of return.

Note that equation (2) together with the lifetime budget constraint (1) represents a system of two equations into two unknowns ( $c_1, c_2$ ) which can therefore be solved to obtain the exact expressions for consumption in both periods.

**Remark 1** *It is useful to define the elasticity of intertemporal substitution EIS, as the elasticity of the ratio ( $c_2/c_1$ ) to a change in the marginal rate of substitution  $MRS_{c_2, c_1} = \beta u'(c_2)/u'(c_1)$ , or*

$$EIS = \frac{d \ln(c_2/c_1)}{d \ln \beta [u'(c_2)/u'(c_1)]}$$

*This is straightforward to interpret. Compute the percentage change in the ratio of marginal utility at 1 and 2 that one percent change in the ratio of consumption at the same dates lead to. The inverse of the number is the intertemporal elasticity of substitution. Graphically, in the  $(c_1, c_2)$  space, the (inverse of the) EIS tells us how the slope of the indifference curve changes following a change in the  $c_2/c_1$  ratio, i.e. following a change in the slope of the radius going through the origin.*

**Example 1** *Suppose households have preferences*

$$U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

*which are called **CES** (constant intertemporal elasticity of substitution) preferences, where  $\gamma \in [0, +\infty)$  and the (constant) elasticity of substitution is  $1/\gamma$ . It is easy to see that this preference specification leads to the Euler equation*

$$\frac{c_2}{c_1} = [\beta(1+r)]^{\frac{1}{\gamma}}.$$

The Euler equation makes it clear that consumption allocation between time 1 and time 2 is determined by three forces:

- **Patience:** related to the discount factor  $\beta$ . The lower is  $\beta$  the lower is the degree of patience of households: households with low  $\beta$  are impatient and they prefer to consume immediately, i.e.  $c_1 \gg c_2$ . For example, in the extreme case where  $\beta = 0$  agent consume all their income in period 1 and  $c_2 = 0$  as they do not value the future.
- **Interest rate:** related to  $r$ . The higher is  $r$  the more profitable it is to save, earn a high return and postpone consumption until the next period.
- **Consumption smoothing:** related to the concavity of  $U$  and hence to the elasticity of substitution  $1/\gamma$ . The term  $\frac{1}{\gamma}$  measures how willing individuals are to substitute intertemporally between consumption this period and consumption next period. It is the inverse of a measure of concavity: for  $\gamma = 0$ ,  $u$  is linear and as  $\gamma$  grows the function becomes more and more concave. For a given combination of patience and return  $\beta(1+r)$ , the lower is  $\gamma$ , the higher is willing to substitute and the steeper is the consumption profile over the two periods: a large EIS magnifies the combined effects of  $(\beta, r)$ .

Another interpretation of the EIS is that it measures the sensitivity of consumption growth to changes in the interest rate (the return of investment opportunities). In our CES example, if we define  $g_c$  as the growth rate of consumption  $(c_{t+1} - c_t)/c_t$ , we have that

$$1 + g_c = [\beta(1+r)]^{\frac{1}{\gamma}},$$

so if we take logarithms of both sides and use the approximation  $\ln(1+x) \simeq x$ , we get

$$g_c = \frac{1}{\gamma} \ln \beta + \frac{1}{\gamma} \ln(1+r) \simeq \frac{1}{\gamma} \ln \beta + \frac{1}{\gamma} r$$

Suppose  $\beta = 1$  so  $\ln \beta = 0$ ; With  $\gamma = .5$ , when  $r$  increases from 3% to 6%,  $g_c$  increases from 6% to 12%. With  $\gamma = 3$ , then the same rise in the interest rate implies that  $g_c$  goes from 1% to 2%, a much smaller increase in the growth rate of consumption.

Finally, note that a steeper consumption profile means more savings in period one to finance a high consumption in period 2.

## 1.2 Interest rate as a relative price

The Euler Equation

$$\frac{u'(c_1)}{\beta u'(c_2)} = (1+r)$$

describes the optimal intertemporal consumption/saving choice of the household. When we studied the static economy we interpreted the optimal static leisure/labor supply choice of the households graphically in the  $(c, l)$  space, in terms of the equality between  $MRS_{c,l}$  and the relative price of leisure, i.e. the wage. Can we give a similar

interpretation to the Euler equation? Yes, we can, if we use the different Cartesian space  $(c_1, c_2)$ . Note from (1) that the slope of the budget constraint is  $-(1+r)$  and the  $MRS_{c_1, c_2}$  between current and future consumption equals  $\frac{u'(c_1)}{\beta u'(c_2)}$ .

It follows that  $(1+r)$  has to be the relative price of consumption this period in terms of consumption next period. To understand, consider again the budget constraints for the two periods, and denote with  $p_1$  the price of the consumption good in period 1, and with  $p_2$  the price of the consumption good in period 2. Then, we have

$$\begin{aligned} p_1 c_1 + p_1 s &= p_1 y_1 \\ p_2 c_2 &= p_1 s + p_2 y_2 \end{aligned}$$

Now, with some simple algebra (substituting  $s$  out of the first budget constraint), we arrive at

$$\begin{aligned} p_2 c_2 &= p_1 (y_1 - c_1) + p_2 y_2 \\ c_2 &= \frac{p_1}{p_2} (y_1 - c_1) + y_2 \end{aligned}$$

Now, if we define  $(1+r) \equiv \frac{p_1}{p_2}$ , we arrive immediately at the lifetime budget constraint formulated in (1).

So the **interest rate is the (intertemporal) price of the consumption good this period, in terms of next period consumption good**. A high interest rate means that there is a high return from postponing consumption to next period, since the price of consumption will be lower than in the current period, i.e., the same dollar amount is more valuable.

### 1.3 Income and Substitution Effect for Borrowers and Lenders

We want to study what happens to savings  $s$  after an increase in the interest rate  $r$ , the relative price of consumption today relative to consumption tomorrow. The change in  $r$  tilts the budget constraint and induces an income and a substitution effect. We need to distinguish three cases.

1. Household with no savings ( $c_1 = y_1$  and  $c_2 = y_2$ ): There is no income effect, so an increase in the interest rate leads only to an intertemporal substitution effect which induces to decrease  $c_1$  and increase savings, so the household will start saving a positive amount  $s > 0$ .
2. Household with positive wealth ( $c_1 < y_1$  and  $c_2 > y_2$ ): There is an income effect that induces the individual to consume more at  $t = 1$  and save less, as she's richer: she earns more interest income on the same wealth. The substitution effect pushes in the opposite direction, as it creates an extra incentive to save. Effect on  $c_1$  and savings  $s$  is ambiguous, but  $c_2$  increases unambiguously.

3. Household with debt ( $c_1 > y_1$  and  $c_2 < y_2$ ): The income effect is negative, the individual is poorer (must pay higher interest on her debt) and would like to cut consumption and reduce her debt (i.e., save more), so the income effect pushes in the same direction as the substitution effect, the individual borrows less and consumes less in the first period, so  $c_1$  falls. The effect on  $c_2$  is ambiguous.

We summarize the income and substitution effects on  $(c_1, c_2)$  in these three cases in the following Table

	No Debt ( $s = 0$ )		Lender ( $s > 0$ )		Borrower ( $s < 0$ )	
	$c_1$	$c_2$	$c_1$	$c_2$	$c_1$	$c_2$
Substitution	↓	↑	↓	↑	↓	↑
Income	0	0	↑	↑	↓	↓
Total	↓	↑	?	↑	↓	?

## 1.4 Credit Market Imperfections I: borrowing constraints

Suppose that households can only borrow up to an amount  $b$ , i.e., they face the constraint  $s \leq -b$ . The value of  $b > 0$  is taken to be a parameter of the problem. Then, given the period one and period two budget constraints

$$\begin{aligned} c_1 + s &= y_1 \\ c_2 &= s(1+r) + y_2 \end{aligned}$$

it follows that we can rewrite the borrowing constraint as

$$c_1 \leq y_1 + b$$

or the household can consume in excess of current income only up to  $b$ . By plotting this additional constraint, together with the budget set, in the  $(c_1, c_2)$  space, it is immediate to see that the new budget set has a kink where the budget line with slope  $-(1+r)$  meets the vertical line at  $c_1 = y_1 + b$ .

How do we solve for the optimal consumption/saving decision in the presence of borrowing constraints? First, solve the “relaxed problem” where you assume the constraint is not binding. Call this candidate solution  $(\tilde{c}_1, \tilde{c}_2)$  and verify that  $\tilde{c}_1 \leq y_1 + b$ . If that is true, then we have found the solution, i.e.,  $(\tilde{c}_1, \tilde{c}_2) = (c_1^*, c_2^*)$  and we are done since the constraint is not binding. If, instead,  $\tilde{c}_1 > y_1 + b$ , then the constraint is binding so the best the household can do is borrow up to the limit and set  $c_1^* = y_1 + b$  and  $c_2^* = -(1+r)b + y_2$ .

## 1.5 Credit Market Imperfections II: wedge between borrowing and saving rate

Suppose that credit markets are imperfect and intermediaries (like banks) make a positive profit from intermediation, so that there is a wedge between the interest rate

gained by lenders and that paid by borrowers, or  $r^L < r^B$ . What is the effect of this wedge on the optimal choice of the household?

The period one budget constraint is unchanged, or

$$c_1 + s = y_1.$$

The period two budget constraint becomes now

$$\begin{aligned} c_2 &= s(1 + r^s) + y_2, \text{ if } s \geq 0, \\ c_2 &= s(1 + r^b) + y_2, \text{ if } s < 0. \end{aligned}$$

So, it is easy to see that, once we derive the intertemporal budget constraint and we graph it in the  $(c_1, c_2)$  space, it displays a kink at the endowment point  $(y_1, y_2)$ . The kink has the implication that small changes in the interest rate have no effect on the optimal saving choice, something unusual.

How do we determine the optimal choice in the case when there is this wedge between borrowing and saving rates? We use a “guess and verify” approach. Follow 3 steps:

1. Assume the individual is a lender, so suppose the Euler equation holds with  $r^s$  or

$$\frac{u'(c_1^*)}{\beta u'(c_2^*)} = (1 + r^s).$$

Using the budget constraint, solve for  $c_1^*$  and verify that  $c_1^* < y_1$ . If it is true, you have found the solution. If it is not true, go to step 2.

2. Assume the individual is a borrower, so suppose the Euler equation holds with  $r^b$  or

$$\frac{u'(c_1^*)}{\beta u'(c_2^*)} = (1 + r^b).$$

Using the budget constraint, solve for  $c_1^*$  and verify that  $c_1^* > y_1$ . If it is true, you have found the solution. If not true, go to step 3.

3. The only option left is that the individual is at the kink and  $c_t^* = y_t^*$ ,  $t = 1, 2$ .