PROBLEM SET 7

THE ECONOMY

- two consumers, 1 and 2.
- Two goods: A and X, written in this order.
- One firm, with production function

$$\hat{A} = F(\hat{X}) = \begin{cases} 0 & \text{if} \quad \hat{X} \leq 1 \\ \hat{X} - 1 & \text{if} \quad \hat{X} > 1 \end{cases} \tag{1}$$

- Consumer 1
  Consumption set $\mathbb{R}^2_+$
  utility function $U_1 = A_1 X_1$
- Consumer 2
  Consumption set $\mathbb{R}^2_+$
  utility function $U_2 = X_2$
- Aggregate endowment vector $[0, 2]$

1.compute all efficient allocations

2.Compute all competitive equilibria with transfers.

3.Does the first welfare theorem hold true?

4.Does the second welfare theorem hold true?

ANSWERS

EFFICIENT ALLOCATIONS

The feasible set is

$$S = \{[A_1, X_1, X_2, \hat{X}] \in \mathbb{R}^4_+ : A_1 \leq F(\hat{X}), X_1 + X_2 + \hat{X} \leq 2\} \tag{2}$$

The objective function is

$$\mathbb{R}^4_+ \xrightarrow{f} \mathbb{R}^2, f(A_1, X_1, X_2, \hat{X}) = [A_1 X_1, X_2] \tag{3}$$

The point $Z_2 = [A_1 = 0, X_1 = 0, X_2 = 2, \hat{X} = 0]$ is efficient. Any efficient point with $\hat{X} \leq 1$ is Z_2 .Hence from now on we restrict the search for efficient points to those that satisfy $\hat{X} > 1$.The feasible set becomes

$$S=\{[A_1,X_1,X_2,\hat{X}]\in\mathbb{R}^4_+ : A_1\le\hat{X}-1, X_1+X_2+\hat{X}\le 2, \hat{X}\ge 1\} \tag{4}$$

We write down the first auxiliary problem, for $\theta=[\theta_1,\theta_2]\in\mathbb{R}^2$

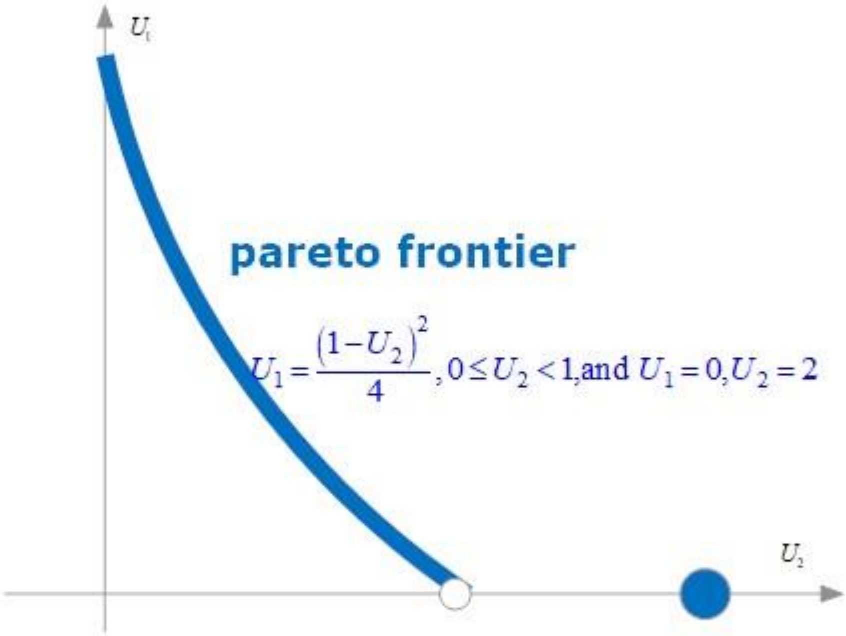
$$\boxed{\begin{array}{l} \underline{P_1(\theta)} \\ \text{MAX } U_1 = A_1X_1 \text{ subject to} \\ A_1\le\hat{X}-1 \\ X_1+X_2+\hat{X}\le 2 \\ \hat{X}\ge 1 \\ X_2\ge\theta_2 \\ [A_1,X_1,X_2,\hat{X}]\in\mathbb{R}^4_+ \\ \text{variables } A_1,X_1,X_2,\hat{X} \\ \text{parameters } \theta_2 \end{array}} \tag{5}$$

We record the global maxima of (5)

$$G_1(\theta)=\begin{cases} [A_1=\frac{1-\theta_2}{2}, X_1=\frac{1-\theta_2}{2}, X_2=\theta_2, \hat{X}=\frac{3-\theta_2}{2}] & \text{if } 0\le\theta_2<1 \\ [A_1=\frac{1}{2}, X_1=\frac{1}{2}, X_2=\theta_2, \hat{X}=\frac{3}{2}] & \text{if } \theta_2\le 0 \\ \varnothing & \text{if } \theta_2\ge 1 \end{cases} \tag{6}$$

By (6),the maxima are essentially unique, hence we do not have to solve the auxiliary problem of consumer 2,and the set of efficient points is $\{Z_2\}\cup\bigcup_{\theta\in\mathbb{R}^2} G_1(\theta)$ .Eliminating θ we obtain

$$\boxed{\begin{array}{l} \underline{\text{efficient points}} \\ \text{EFFICIENT POINTS}=\{Z_2\}\cup \\ \bigcup_{0\le X_2<1}\left\{[A_1=\frac{1-X_2}{2}, X_1=\frac{1-X_2}{2}, X_2, \hat{X}=\frac{3-X_2}{2}]\right\} \end{array}} \tag{7}$$



COMPETITIVE EQUILIBRIA WITH TRANSFERS

We compute competitive equilibria for all endowment vectors $e_1 = [0, \bar{X}_1] \in \mathbb{R}^2_+, e_2 = [0, \bar{X}_2] \in \mathbb{R}^2_+$ such that $e_1 + e_2 = [0, 2]$,and all profit share parameters σ_1, σ_2

1. *NAME the price of each good*

$p = price\ of\ A, w = price\ of\ X$. Normalize $p = 1$

2.DEFINE consumer incomes

$$m_i = w\bar{X}_i + \sigma_i \Pi, i = 1..2 \tag{8}$$

3. SOLVE the optimization problem of the firm

profit $\Pi = pF\left(\hat{X}\right) - w\hat{X} = \begin{cases} -w\hat{X} & \text{if} \quad \hat{X} \le 1 \\ (1-w)\hat{X} - 1 & \text{if} \quad \hat{X} \ge 1 \end{cases}$

is maximized at

$$[\hat{A}, \hat{X}, \Pi] = \begin{cases} \text{NONE} & \text{if} \quad w < 1 \\ [0, 0, 0] & \text{if} \quad w \ge 1 \end{cases} \tag{9}$$

By (9),the only candidate equilibrium prices are $w \ge 1$ .Hence from now on

$$[\hat{A}, \hat{X}, \Pi] = [0, 0, 0], w \ge 1, m_i = w\bar{X}_i \tag{10}$$

4. SOLVE the optimization problems of consumers

$\max\ U_2 = X_2$ subject to

$0 \le wX_2 \le m_2 = w\overline{X}_2$

variables : X_2

$\max\ U_1 = A_1 X_1$

subject to

$A_1 + wX_1 \le m_1 = w\overline{X}_1$

$A_1 \ge 0, X_1 \ge 0$

variables : A_1, X_1

The solutions are

$$X_2 = \overline{X}_2$$
$$[A_1, X_1] = [\frac{w\overline{X}_1}{2}, \frac{\overline{X}_1}{2}] \quad (11)$$

5. SOLVE the equilibrium conditions

$$A_1 = \hat{A}, X_2 + X_1 + \hat{X} = 2 \quad (12)$$

By (12),(11),(10) we obtain

$$\frac{w\overline{X}_1}{2} = 0, \overline{X}_2 + \frac{\overline{X}_1}{2} + 0 = 2 \quad (13)$$

The system (13) has no solutions, unless $\overline{X}_1 = 0, \overline{X}_2 = 2$ hence

$$\boxed{\begin{array}{l} \text{equilibria with transfers} \\ Z_2 = [A_1 = 0, X_1 = 0, X_2 = 2, \hat{X} = 0] \\ \text{EQ}(e_1, e_2, \sigma_1, \sigma_2) = \begin{cases} \{Z_2\} & \text{IF} \quad \overline{X}_2 = 2 \\ \varnothing & \text{otherwise} \end{cases} \end{array}} \quad (14)$$

By (14) we obtain

$$\bigcup_{e_1, e_2, \sigma_1, \sigma_2} \text{EQ}(e_1, e_2, \sigma_1, \sigma_2) = \{Z_2\} \quad (15)$$

3.Does the first welfare theorem hold true?

Yes, because $\bigcup_{e_1, e_2, \sigma_1, \sigma_2} \text{EQ}(e_1, e_2, \sigma_1, \sigma_2) = \{Z_2\} \subseteq \text{EFFICIENT POINTS}$

4.Does the second welfare theorem hold true?

No, because $\text{EFFICIENT POINTS} \not\subset \bigcup_{e_1,e_2,\sigma_1,\sigma_2} \text{EQ}(e_1,e_2,\sigma_1,\sigma_2) = \{Z_2\}$

In this example, there is a tradeoff between efficiency and distribution. The only efficient point that is decentralizable is Z_2 ,where consumer 2 gets everything and consumer 1 gets nothing. Efficiency cannot be separated from distribution.