PROBLEM SET 6

THE ECONOMY

- N+1 consumers, N>4.
- Two goods: A and X, written in this order.
- One firm, with production function $\hat{A} = 2\sqrt{\hat{X}}, \hat{X} \geq 0$
- Consumer 0 is the sole owner of the firm.
  Consumption set $\mathbb{R}^2_+$
  Endowment vector $e_0 = [0,0]$
  utility function $U_0 = X_0$
- Consumers i=1...N
  Consumption set $\mathbb{R}^2_+$
  Endowment vector $e_i = [0,1]$.(the endowment of good A is zero).
  utility function $U_i = A_i X_i$

Policy measures: Firm profit Π is taxed at a rate $t, 0 < t < 1$.The tax revenue $t\Pi$ is distributed equally to consumers 1,2,..,N with a lump-sum transfer
$T_i = \frac{t\Pi}{N}, i = 1...N$

1.compute all efficient allocations EFF(N) of this economy as a function of population size N. (The linear SWF method works in this example)

2.Compute all competitive equilibrium allocations E(t,N) of this economy as a function of the tax rate t and population size N.

3.For which values of the parameters (t,N), if any, is E(t,N) a subset of EFF(N) ?

ANSWERS

1.compute all efficient allocations EFF(N) of this economy as a function of population size N

The efficiency problem of this economy is defined by $S \xrightarrow{f} \mathbb{R}^{N+1}$,where

$$S = \text{set of feasible allocations} = \left\{ \begin{array}{l} [X_0, X_1, ..., X_N, A_1, ..., A_N, \hat{A}, \hat{X}] \in \mathbb{R}_+^{2N+3} \text{ such that} \\ A_1 + ... + A_N \leq \hat{A}, X_0 + X_1 + ... + X_N + \hat{X} \leq N, \hat{A} = 2\sqrt{\hat{X}} \end{array} \right\} \quad (1)$$

Eliminating $\hat{A}$,the feasible set becomes

$$S=\left\{\begin{matrix}[X_0,X_1,...,X_N,A_1,...,A_N,\hat{X}]\in\mathbb{R}_+^{2N+2} \text{ such that} \\ A_1+...+A_N\le 2\sqrt{\hat{X}}, X_0+X_1+...+X_N+\hat{X}\le N\end{matrix}\right\} \tag{2}$$

Note that the feasible set is convex, as the intersection of the better-than sets of concave functions.

The objective function is the vector of the utility functions.

$$f=[U_0,U_1,...,U_N]=[X_0,A_1X_1,...,A_NX_N] \tag{3}$$

Since pareto efficiency is an ordinal concept and log is a strictly increasing function, we obtain the same efficient allocations if we use the objective function

$$\begin{aligned} g&=[g_0,g_1,...,g_N]=[\log U_0,\log U_1,...,\log U_N] \\ &=[\log X_0,\log A_1+\log X_1,...,\log A_N+\log X_N] \end{aligned} \tag{4}$$

Note that each utility function g_i is strictly concave. (This trick only works if the original utility function U_i is quasi-concave).

By (4) and the convexity of S, the linear SWF method of computing efficient points is complete. By strict monotonicity of the utility functions and the absence of externalities, every weakly efficient point is efficient, and therefore the linear SWF method of computing efficient points is sound.

$$\boxed{\text{the linear SWF method is sound and complete for this particular problem}} \tag{5}$$

$$\boxed{\begin{array}{l} \boxed{\text{linear SWF}} \\ W_\theta=\theta_0g_0+\theta_1g_1+...+\theta_Ng_N \\ \theta_i\ge 0 \text{ for all } i\ge 0, \theta_0+\theta_1+...+\theta_N=1 \end{array}} \tag{6}$$

We solve the maximization problem $[W_\theta,S]$ for all values of $\theta=[\theta_0,...,\theta_N]$ as in (6).

$$\begin{array}{|l|}\hline \max\, W_\theta = \theta_0 \log X_0 + \sum_{i=1}^{N} \theta_i \log A_i + \sum_{i=1}^{N} \theta_i \log X_i \\ \text{subject to} \\ A_1 + \ldots + A_N \le 2\sqrt{\hat{X}} \\ X_0 + X_1 + \ldots + X_N + \hat{X} \le N \\ [X_0, X_1, \ldots, X_N, A_1, \ldots, A_N, \hat{X}] \in \mathbb{R}_+^{2N+2} \\ \text{Variables:} X_0, X_1, \ldots, X_N, A_1, \ldots, A_N, \hat{X} \\ \text{parameters:N}, \theta_0, \ldots, \theta_N \\ \text{Conditions on parameters:} \\ \text{N>4}, \theta_0 \ge 0, \ldots, \theta_N \ge 0, \sum_{i=0}^{N} \theta_i = 1 \\ \hline \end{array} \tag{7}$$

By (5),the efficient points are the global maxima of (7),namely

$$\begin{array}{l} \underline{\mathrm{EFF}_1|} \\ \theta_0 = 1 \Rightarrow \\ X_0 = N, \hat{X} = A_i = X_i = 0, i \ge 1 \\ \theta_k = 1, k \ge 1 \Rightarrow \\ A_i = X_i = 0, i \ne k, \hat{X} = N/3, A_k = 2\sqrt{N/3}, X_k = 2N/3 \end{array} \tag{8}$$

$$\begin{array}{l} \underline{\mathrm{EFF}_2(I), I \subseteq \{1,..,N\}|} \\ \theta_0 = 0, \theta_i > 0 \text{ iff } i \in I \Rightarrow \\ X_0 = 0, \hat{X} = N/3 \\ X_i = A_i = 0, i \notin I \\ X_i = A_i\sqrt{N/3}, i \in I \\ \sum_{i \in I} A_i = 2\sqrt{N/3} \end{array} \tag{9}$$

$$\begin{array}{l} \underline{\mathrm{EFF}_3|} \\ \theta_i > 0, \text{all } i \Rightarrow \\ X_0 = N - 3\hat{X}, 0 < \hat{X} < N/3 \\ X_i = A_i\sqrt{\hat{X}} \\ \sum_{i=1}^{N} A_i = 2\sqrt{\hat{X}} \end{array} \tag{10}$$

$$\begin{array}{l}
\underline{\mathrm{EFF}_4(I),\ I \subseteq \{1,..,N\}|} \\
\theta_i > 0 \text{ iff } i \in I, \theta_0 > 0 \Rightarrow \\
X_0 = N - 3\hat{X}, 0 < \hat{X} < N/3 \\
X_i = A_i = 0, i \notin I \\
X_i = A_i\sqrt{\hat{X}}, i \in I \\
\sum_{i \in I} A_i = 2\sqrt{\hat{X}}
\end{array} \tag{11}$$

We conclude that

$$\boxed{EFF(N) = EFF_1 \cup EFF_3 \cup \bigcup_I EFF_2(I) \cup \bigcup_I EFF_4(I)} \tag{12}$$

2.Compute all competitive equilibrium allocations E(t,N) of this economy as a function of the tax rate t and population size N

1. NAME the price of each good

$p = price\ of\ A, w = price\ of\ X$. Normalize $w = 1$

2.DEFINE consumer incomes

$$m_0 = (1-t)\Pi, m_i = 1 + \frac{t\Pi}{N}, i \geq 1 \tag{13}$$

3. SOLVE the optimization problems of the firm

After-tax profit $(1-t)\Pi = (1-t)\left(p\hat{A} - \hat{X}\right) = (1-t)\left(2p\sqrt{\hat{X}} - \hat{X}\right)$ is maximized at

$$\begin{bmatrix} \hat{A} \\ \hat{X} \\ \Pi \end{bmatrix} = \begin{bmatrix} 2p \\ p^2 \\ p^2 \end{bmatrix} \tag{14}$$

4. SOLVE the optimization problems of consumers

$$\begin{array}{l}
\max\ U_0 = X_0 \\
\text{subject to} \\
0 \leq X_0 \leq m_0 \\
\text{variables}: X_0
\end{array}$$

$$\begin{array}{l}
\max\ U_i = A_i X_i \\
\text{subject to} \\
pA_i + X_i \leq m_i, A_i \geq 0, X_i \geq 0 \\
\text{variables}: A_i, X_i
\end{array}$$

The solutions are

$$X_0 = m_0$$
$$[A_i, X_i] = [\frac{m_i}{2p}, \frac{m_i}{2}], i \geq 1 \tag{15}$$

5. SOLVE the equilibrium conditions

$$A_1 + \ldots + A_N = \hat{A}$$
$$X_0 + X_1 + \ldots + X_N + \hat{X} = N \tag{16}$$

There is a unique solution, described by

$$\boxed{\begin{array}{l} \text{equilibrium price and allocation} \\ p = \sqrt{\frac{N}{4-t}} \\ \hat{A} = 2\sqrt{\frac{N}{4-t}}, \hat{X} = \Pi = \frac{N}{4-t} \\ U_0^E = X_0 = m_0 = \frac{N(1-t)}{4-t} \\ m_i = \frac{4}{4-t}, i \geq 1 \\ [A_i, X_i] = [\frac{2}{\sqrt{N(4-t)}}, \frac{2}{4-t}] \\ U_i^E = \frac{4}{N^{1/2}(4-t)^{3/2}}, i \geq 1 \end{array}} \tag{17}$$

By (17),the set of equilibrium allocations is

$$\boxed{E(t,N) = \left\{ \begin{array}{l} [X_0, X_1, \ldots, X_N, A_1, \ldots, A_N, \hat{X}] \in \mathbb{R}_+^{2N+2} \text{ such that} \\ \hat{X} = \frac{N}{4-t}, X_0 = \frac{N(1-t)}{4-t} \\ [A_i, X_i] = [\frac{2}{\sqrt{N(4-t)}}, \frac{2}{4-t}], i \geq 1 \end{array} \right\}} \tag{18}$$

3.For which values of the parameters (t,N), if any, is E(t,N) a subset of EFF(N) ?

$E(t,N) \subseteq \mathrm{EFF}_3 \subseteq \mathrm{EFF}(N)$,by comparing (18) to (10).