PS4,answers

Wednesday, March 20, 2024 10:48

PROBLEM SET 4

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $U(x,y) = x + \theta \log(y+A)$

constraints $x + py \leq m, x \geq 0, y \geq 0$

variables x, y

parameters θ, A, p, m

conditions on parameters $\theta > 0, A > 0, p > 0, m > 0$

STEP 1 NORMAL FORM

MAX U

$m - x - py \geq 0, \quad x \geq 0, \quad y \geq 0$

STEP 2: LAGRANGIAN

$$L = \lambda_0 \cdot U + \lambda_1 (m - x - py)$$

STEP 3: WHICH THEOREMS APPLY?

ALL THREE

STEP 4: FRITZ JOHN NECESSARY CONDITIONS

$\boxed{1_x}$ $\frac{\partial L}{\partial x} = \lambda_0 - \lambda_1 \leq 0$, $\boxed{2_x}$ $\frac{\partial L}{\partial x} \cdot x = 0$

$\boxed{1_y}$ $\frac{\partial L}{\partial y} = \frac{\lambda_0 \theta}{y + A} - \lambda_1 p \leq 0$, $\boxed{2_y}$ $\frac{\partial L}{\partial y} \cdot y = 0$

$\boxed{3}$ $m - x - py \geq 0$ $\boxed{4}$ $(m - x - py)\lambda_1 = 0$

$\boxed{13}$ $m - x - py \geq 0$ $\boxed{14}$ $(m - x - py) \cdot \lambda_1 = 0$

$\boxed{15}$ $x, y \geq 0$ $\boxed{16}$ $\lambda_0, \lambda_1 \geq 0$

$\boxed{17}$ $\lambda_0 = 0$ OR 1, NOT ALL $\lambda_j = 0$

STEP 5: SEARCH FOR SOLUTIONS OF NECESSARY CONDITIONS

HYPOTHESIS: $\lambda_0 = 1, \; x > 0, \; y > 0$

SOLUTION: BY $\boxed{Ex}$, $\lambda_1 = 1$. THEN BY $\boxed{14}$

$$x + py = m$$

BY $\boxed{Ey}$ $\dfrac{\theta}{y+A} = p$ HENCE

$$y + A = \frac{\theta}{p} \qquad y = \frac{\theta}{p} - A$$

$$x + \theta - pA = m, \qquad x = m + pA - \theta$$

CONSISTENCY CHECK:

THE CONDITIONS $x > 0, y > 0$ YIELD

$$\frac{\theta - m}{A} < p < \frac{\theta}{A}$$

SINCE ARROW-ENTHOVEN HOLDS, WE CONCLUDE

GLOBAL MAXIMA

$$\left(\left[m + pA - \theta, \; \frac{\theta}{p} - A \right] \text{ if } \frac{\theta - m}{A} < p < \frac{\theta}{A} \right.$$

$$[x,y] = \begin{cases} [m+pA-\theta,\ \frac{\theta}{p}-A] & \text{if } \frac{\theta-m}{A} < p < \frac{\theta}{A} \\ ? & \text{OTHERWISE} \end{cases}$$

AND WE CONTINUE THE SEARCH FOR SOLUTIONS OF THE NECESSARY CONDITIONS

HYPOTHESIS: $\lambda_0 = 1,\ x > 0,\ y = 0$

SOLUTION: $\lambda_1 = 1,\ x = m,\ y = 0$

CONSISTENCY CHECK

BY $\boxed{1_y}$ $\quad \theta/A \le p$

SINCE ARROW-ENTHOVEN HOLDS, WE CONCLUDE

GLOBAL MAXIMA

$$[x,y] = \begin{cases} [m+pA-\theta,\ \frac{\theta}{p}-A] & \text{if } \frac{\theta-m}{A} < p < \frac{\theta}{A} \\ [m,0] & \text{if } p \ge \theta/A \\ ? & \text{OTHERWISE} \end{cases}$$

AND WE CONTINUE THE SEARCH

HYPOTHESIS: $\lambda_0 = 1,\ x = 0,\ y > 0$

SOLUTION: $\lambda_1 \ge 1,\ y = m/p,\ \frac{\theta}{y+A} = \lambda_1 p$

HENCE $\lambda_1 = \dfrac{\theta}{m+pA}$

HENCE $\lambda_1 = \dfrac{\theta}{m + pA}$

CONSISTENCY CHECK:

$\dfrac{\theta}{m+pA} \geq 1$ ie $p \leq \dfrac{\theta - m}{A}$, $\theta > m$

SINCE ARROW-ENTHOVEN HOLDS, WE CONCLUDE

GLOBAL MAXIMA

$$[x, y] = \begin{cases} \left[m + pA - \theta, \dfrac{\theta}{p} - A\right] & \text{if } \dfrac{\theta - m}{A} < p < \dfrac{\theta}{A} \\ [m, 0] & \text{if } p \geq \theta / A \\ [0, m/p] & \text{if } p \leq \dfrac{\theta - m}{A}, \ \theta > m \end{cases}$$

AT THIS POINT, WE HAVE EXHAUSTED THE PARAMETER SPACE. WE CONTINUE THE SEARCH UNTIL WE EXHAUST THE VARIABLE SPACE

HYPOTHESIS: $\lambda_0 = 1$, $x = 0$, $y = 0$

SOLUTION: $\lambda_1 \geq 1$, $x + py = m$

CONSISTENCY CHECK: $0 = m$, CONTRADICTION

HYPOTHESIS: $\lambda_0 = 0$

SOLUTION: By $\boxed{7}$ $\lambda_1 > 0$, so by $\boxed{4}$

$$x + py = m$$

By $\boxed{1y}$ $\dfrac{\partial L}{\partial y} = -\lambda_1 p < 0$, so by $\boxed{\varepsilon y}$

$$y = 0, \quad x = m > 0.$$

THEN BY $\boxed{\varepsilon x}$ $\dfrac{\partial L}{\partial x} = 0$ ie $0 - \lambda_1 = 0$

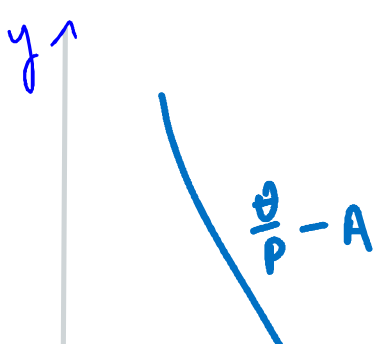
CONSISTENCY CHECK: CONTRADICTION

$(\lambda_1 > 0$ AND $\lambda_1 = 0)$

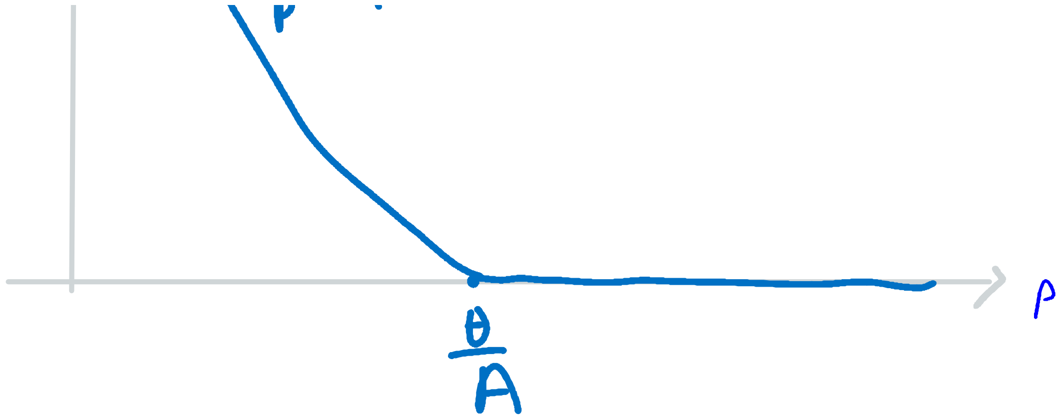
AT THIS POINT WE HAVE EXHAUSTED BOTH THE PARAMETER SPACE AND THE VARIABLE SPACE. WE CONCLUDE

GLOBAL MAXIMA

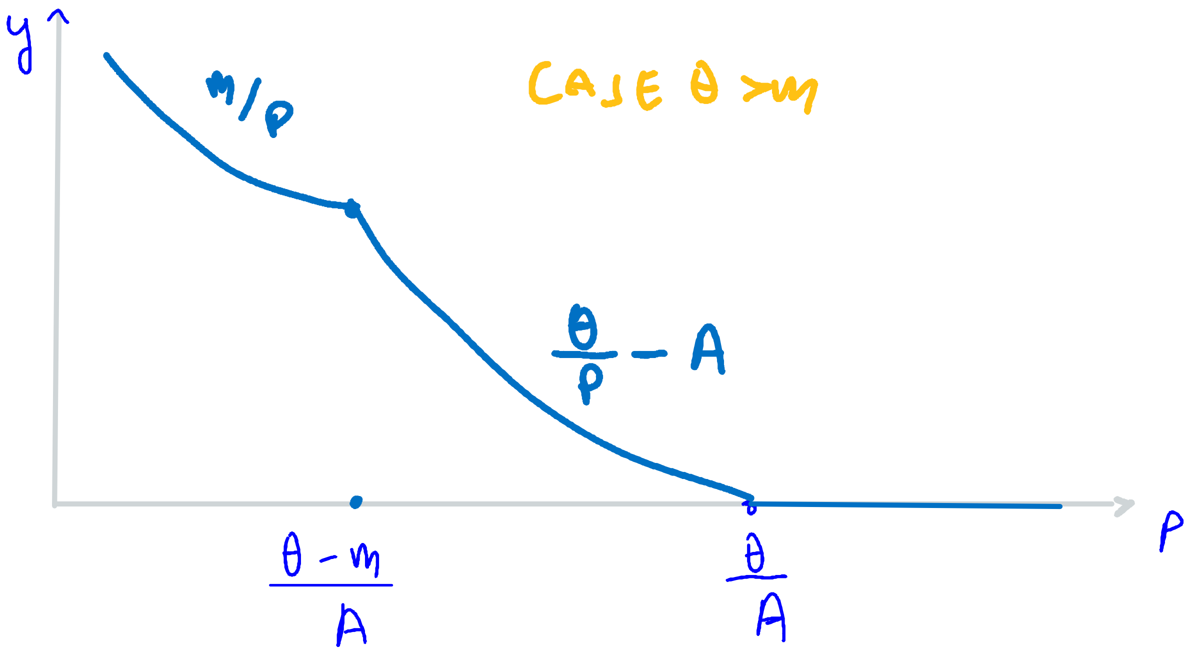
$$[x, y] = \begin{cases} \left[m + pA - \theta, \dfrac{\theta}{p} - A\right] & \text{if } \dfrac{\theta - m}{A} < p < \dfrac{\theta}{A} \\ [m, 0] & \text{if } p \geq \theta / A \\ [0, m/p] & \text{if } p \leq \dfrac{\theta - m}{A}, \ \theta > m \end{cases}$$



CASE $\theta \leq m$



θ
A
p



y
m/p
CASE θ > m
θ/p − A
(θ − m)/A
θ/A
p