PS3,answers

Wednesday, March 13, 2024 11:13

PROBLEM SET 3

Consider a firm with production function

$$\hat{A} = g(\hat{X}) = \begin{cases} 0 & \text{if} \quad \hat{X} \le F \\ \sqrt{2\left(\hat{X} - F\right)} & \text{if} \quad \hat{X} \ge F \end{cases} \qquad \textbf{(1)}$$

$\hat{X}$ **is a variable,** F **is a positive parameter.**

1.Draw the production function, and the corresponding production set. Is the production set convex?

2.Let p be the price of the output, and let w be the price of the input. Draw the average cost curve $AC(\hat{X}) = \frac{w\hat{X}}{p\hat{A}} = \frac{w\hat{X}}{pg(\hat{X})}$

3.For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $pg(\hat{X}) - w\hat{X}$

Constraints $\hat{X} \ge 0$

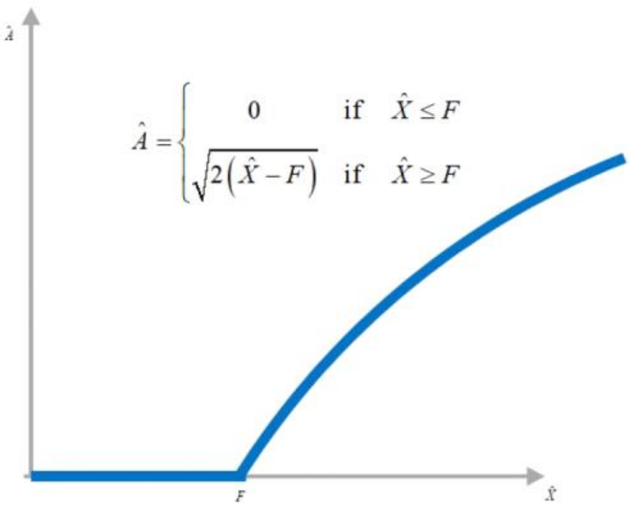
variables $\hat{X}$

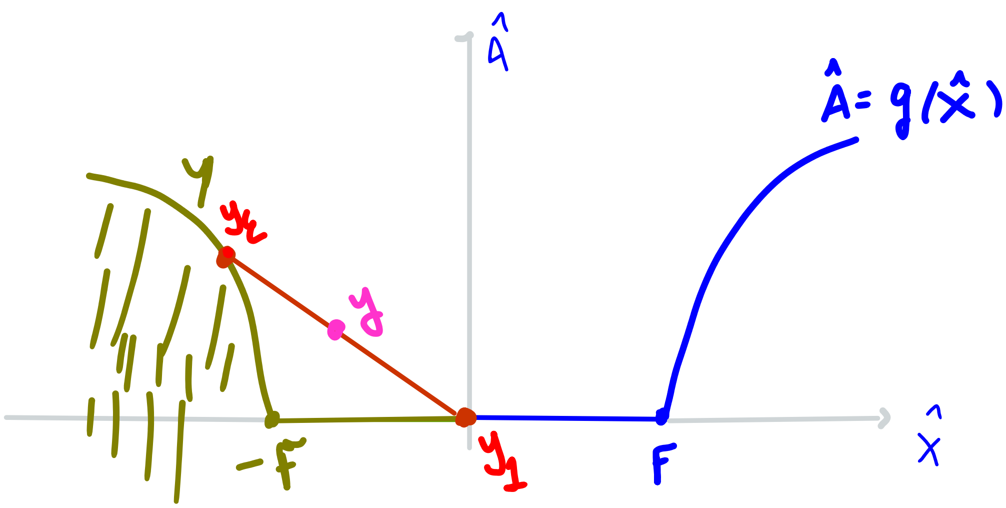
parameters p, w, F

conditions on parameters $p > 0, w > 0, F > 0$

$\boxed{1}$ THE PRODUCTION SET Y CORRESPONDING TO THE PRODUCTION FUNCTION g is

$Y = \{(\hat{A}, -\hat{X}) : \hat{A} \le g(\hat{X}), \hat{X} \ge 0\}$





g IS QUASI-CONCAVE (INCREASING, ONE VARIABLE)

Y IS NOT CONVEX, BECAUSE

$$y_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in Y$$

$$y_2 = \begin{bmatrix} \sqrt{2F} \\ -2F \end{bmatrix} \in Y$$

BUT THE CONVEX COMBINATION $y = \frac{1}{2} y_1 + \frac{1}{2} y_2 \notin Y$

BECAUSE

$$+ \left(\sqrt{F/2} \right) \quad \left[\hat{A} \right]$$

$$y = \begin{bmatrix} \sqrt{F/2} \\ -F \end{bmatrix} = \begin{bmatrix} \hat{A} \\ -\hat{x} \end{bmatrix}, \text{ AND}$$

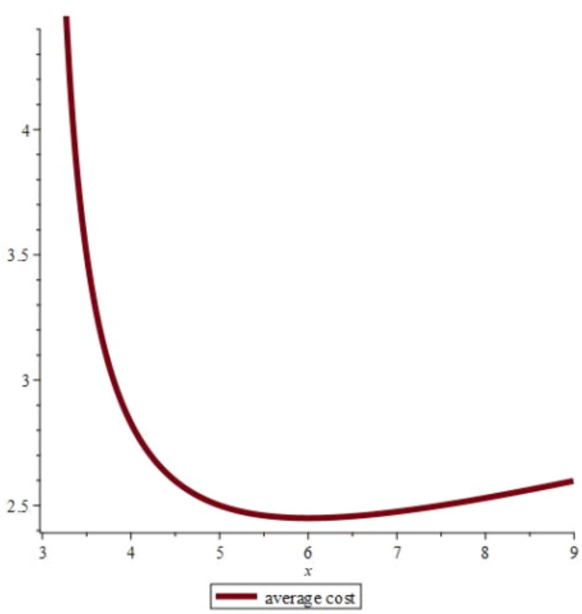
$$g(\hat{x}) = g(F) = 0 < \sqrt{F/2} = \hat{A}$$

$\boxed{2}$

$$AC(\hat{x}) = \begin{cases} \text{UNDEFINED} & \text{if } \hat{x} \le F \\ \dfrac{w}{p} \cdot \dfrac{\hat{x}}{\left(2(\hat{x}-F)\right)^{1/2}} & \text{if } \hat{x} > F \end{cases}$$

$$AC(F) = +\infty = AC(+\infty)$$

$$AC'(\hat{x}) = \frac{\hat{x} - 2F}{\left(2(\hat{x}-F)\right)^{3/2}} = \begin{cases} < 0 & F < \hat{x} < 2F \\ > 0 & \hat{x} > 2F \end{cases}$$



AVERAGE COST IS MINIMIZED AT $\hat{x} = 2F$

$\boxed{3}$ $\max \pi(\hat{x}) = P \cdot g(\hat{x}) - w\hat{x}, \quad \hat{x} \ge 0$

STEP 1: NORMAL FORM OK

STEP 2: LAGRANGIAN, $L = \lambda_0 \cdot \Pi = \pi$

STEP 3: WHICH THEOREMS APPLY ?

WEIERSTRASS NO, FEASIBLE SET NOT BOUNDED

FRITZ JOHN YES, EXCEPT AT $\hat{x} = F$

ARROW-ENTHOVEN NO, BECAUSE THE OBJECTIVE FUNCTION π IS NOT QUASICONCAVE. TO SEE THIS

$$\pi(\hat{x}) = \begin{cases} -w\hat{x} & 0 \le \hat{x} \le F \\ p\sqrt{2(\hat{x}-F)} - w\hat{x}, & \hat{x} > F \end{cases} \qquad (1)$$

$$\pi'(\hat{x}) = \begin{cases} -w & \hat{x} < F \\ \frac{p}{\sqrt{2}}(\hat{x} - F)^{-1/2} - w & \hat{x} > F \end{cases} \qquad (2)$$

$$\pi''(\hat{x}) = \begin{cases} 0 & \hat{x} < F \\ \frac{-p}{2\sqrt{2}}(\hat{x} - F)^{-\frac{3}{2}} & \hat{x} > F \end{cases} \qquad (3)$$

HENCE

$$\pi'(\hat{x}) = \begin{cases} < 0 & \hat{x} < F \\ > 0 & F < \hat{x} < F + \frac{p^2}{2w^2} \\ = 0 & \hat{x} = F + \frac{p^2}{2w^2} \\ < 0 & \hat{x} > F + p^2/2w^2 \end{cases} \qquad (4)$$

$$\pi''(\hat{x}) = \begin{cases} 0 & \hat{x} < F \\ < 0 & \hat{x} > F \end{cases} \qquad (5)$$

BY (4) π IS NEITHER INCREASING, NOR DECREASING, NOR SINGLE-PEAKED, HENCE π IS NOT QUASI CONCAVE.

WE CAN ALSO COMPUTE THE BETTER THAN SET OF π AT $\hat{x} = F/\varepsilon$ (OR EQUIVALENTLY, AT $y = -\frac{wF}{\varepsilon}$)

$$B^{\pi}_{\hat{x}} = \{\hat{x} \in R_+ : \pi(\hat{x}) \geq \pi(F/\varepsilon)\} =$$

$$= \left\{\hat{x} \in R_+ : \pi(\hat{x}) \geq -\frac{wF}{\varepsilon}\right\} =$$

$$\left[0, \frac{F}{2}\right] \cup [t, \infty) = \text{NONCONVEX SET}$$

$$t \triangleq \frac{1}{8} w^2 F^2 + F$$

SINCE ONLY FRITZ JOHN APPLIES FOR $\hat{x} = F$, AND SINCE WE NEED AT LEAST TWO THEOREMS TO SOLVE AN OPTIMIZATION PROBLEM BY THE BOOK, WE NEED TO TRY METHODS APPLICABLE ONLY TO THIS PARTICULAR PROBLEM

$\boxed{3.1}$ SOLVE SEPARATELY

$$\max \pi(\hat{x}) \quad (P1) \qquad\qquad \max \pi(\hat{x}) \quad (P2)$$
$$0 \le \hat{x} \le F \qquad\qquad\qquad \hat{x} \ge F$$

$\boxed{3.2}$ CHOOSE THE BEST OF (P1), (P2)

$\boxed{3.1}$

(P1) IS AN EASY PROBLEM, ALL THEOREMS APPLY

GLOBAL MAXIMA OF (P1)

$$\hat{x} = 0, \quad \pi^1_{max} = 0$$

SOLVING (P2)

STEP 1: NORMAL FORM

$$\max \pi(\hat{x}) = \sqrt{2(\hat{x} - F)} - w\hat{x}$$
$$\hat{x} - F \ge 0, \quad \hat{x} \ge 0$$

STEP 2: LAGRANGIAN

$$L = \lambda_0 \cdot \Pi + \lambda_1(\hat{x} - F)$$

STEP 3: WHICH THEOREMS APPLY?

WEIERSTRASS NO

FRITZ JOHN: YES

ARROW-ENTHOVEN YES, BECAUSE THE OBJECTIVE FUNCTION IS CONCAVE, AND THE FEASIBLE SET CONVEX

STEP 4: WRITE DOWN THE FOCS

$$\frac{\partial L}{\partial \hat{x}} = \lambda_0 \frac{\partial \Pi}{\partial \hat{x}} + \lambda_1 \leq 0, \quad \frac{\partial L}{\partial \hat{x}} \cdot \hat{x} = 0$$

$$\frac{\partial L}{\partial \lambda_1} = \hat{x} - F \geq 0, \quad \frac{\partial L}{\partial \lambda_1} \cdot \lambda_1 = 0$$

$$\hat{x} \geq 0, \quad \lambda_0, \lambda_1 \geq 0$$

$$\lambda_0 = 0 \text{ OR } 1, \quad \text{NOT ALL } \lambda_j = 0$$

STEP 5: SEARCH FOR SOLUTIONS OF THE FOCS

SINCE $\Pi''(\hat{x}) < 0 \;\; \forall \hat{x} > F$, Π IS STRICTLY CONCAVE, HENCE GLOBAL MAXIMA ARE UNIQUE, IF THEY EXIST.

SINCE ARROW-ENTHOVEN APPLIES, ANY SOLUTION OF THE FOCS WITH $\lambda_0 = 1$ IS A GLOBAL MAX

HENCE WE TRY FIRST TO SOLVE THE FOCS

WITH $\lambda_0 = 1$

HYPOTHESIS: $\lambda_0 = 1$, $\hat{x} > F$

SOLUTION: $\lambda_1 = 0$, $\frac{\partial \Pi}{\partial \hat{x}} = 0$ i.e BY (4)

$$\lambda_1 = 0, \quad \hat{x} = F + \frac{p^2}{2w^2}$$

CONSISTENCY CHECK OK

GLOBAL MAXIMA OF (P2)

$$\hat{x} = F + \frac{p^2}{2w^2}, \quad \Pi^2_{MAX} = \frac{p^2}{2w} - wF$$

$\boxed{3.E}$

COMPARE Π^1_{MAX} TO Π^2_{MAX} AND CHOOSE THE BEST

$$\Pi^2_{MAX} > \Pi^1_{MAX} \iff \frac{p^2}{2w} - wF > 0 \iff$$

$$\frac{p^2}{2w^2} > F \iff \frac{p}{w} > \sqrt{2F}$$

HENCE

$p/w > \sqrt{2F} \Rightarrow$ CHOOSE THE GLOBAL MAXIMA OF (P2)

$p/w < \sqrt{2F} \Rightarrow$ CHOOSE THE GLOBAL MAXIMA OF (P1)

$p/w = \sqrt{2F} \Rightarrow$ CHOOSE BOTH

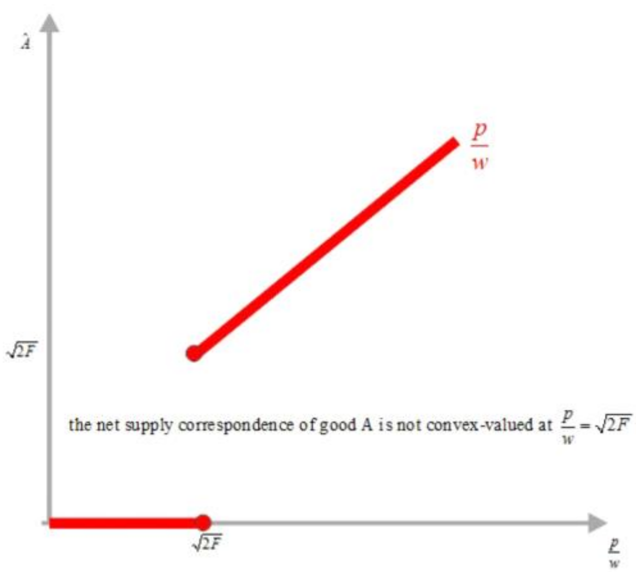
HENCE THE GLOBAL MAXIMA OF THE ORIGINAL

PROFIT MAX PROBLEM ARE

$$
\left(\hat{A},\hat{X},\Pi\right)=\begin{cases} (0,0,0) & \text{if} \quad \frac{p}{w}<\sqrt{2F} \\ \{(0,0,0),(\sqrt{2F},2F,0)\} & \text{if} \quad \frac{p}{w}=\sqrt{2F} \\ (\frac{p}{w},F+\frac{1}{2}\left(\frac{p}{w}\right)^2,\frac{p^2}{2w}-wF) & \text{if} \quad \frac{p}{w}>\sqrt{2F} \end{cases}
$$

THE NET SUPPLY CORRESPONDENCE IS THEN

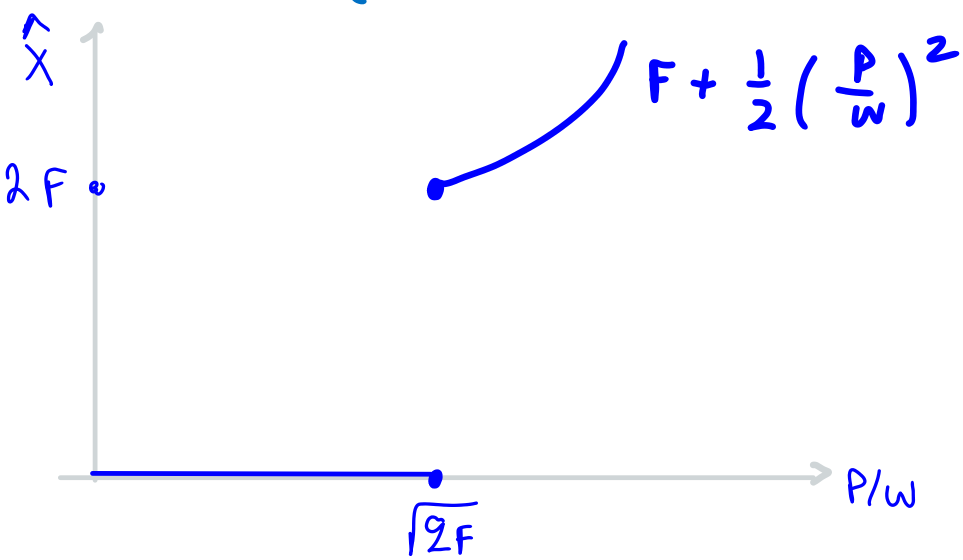
$$
\hat{A} = \begin{cases} 0 & p/w < \sqrt{2F} \\ \{0, \sqrt{2F}\} & p/w = \sqrt{2F} \\ p/w & p/w > \sqrt{2F} \end{cases}
$$



THE INPUT DEMAND CORRESPONDENCE IS THEN

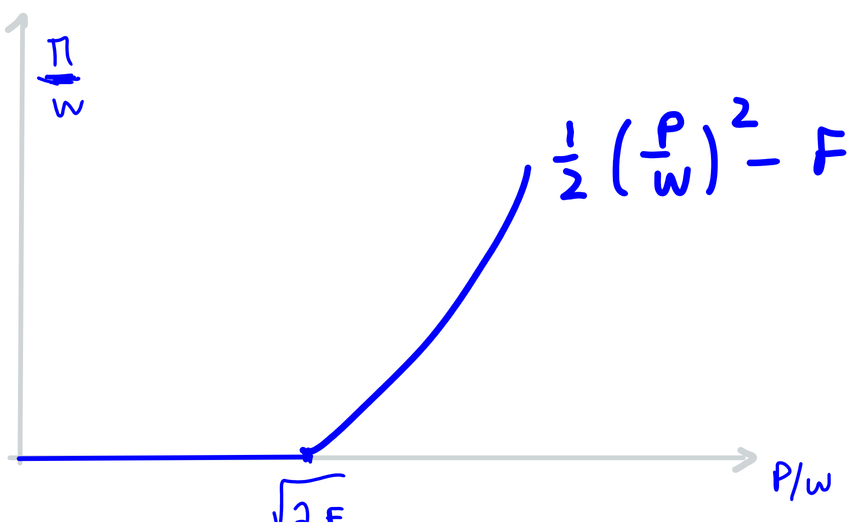
$$
\hat{X} = \begin{cases} 0 & p/w < \sqrt{2F} \\ \{0, 2F\} & p/w = \sqrt{2F} \end{cases}
$$

$$\hat{x} = \begin{cases} 0 & p/w < \sqrt{2F} \\ \{0, 2F\} & p/w = \sqrt{2F} \\ F + \frac{1}{2}\left(\frac{p}{w}\right)^2 & p/w > \sqrt{2F} \end{cases}$$



THE INDIRECT PROFIT FUNCTION IS

$$\frac{\pi}{w} = \begin{cases} 0 & p/w \leq \sqrt{2F} \\ \frac{p^2}{2w^2} - F & p/w > \sqrt{2F} \end{cases}$$



$\sqrt{2F}$
P/w