Takehomeexam2023june,answers

Saturday, June 17, 2023 23:06

takehomeexam2023june

PROBLEM 1

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be an injective linear map and A a closed set in $\mathbb{R}^n$.

1.Show that $T(A)$ is a closed set.

2.Provide an example of a linear map T and a closed set A such that $T(A)$ is not closed

3.Generalize to injective affine maps

PROBLEM 2

Let A be the line in $\mathbb{R}^n$ through the points [1,0,0], [0,1,0]. Derive an explicit formula for the nearest point map $\mathbb{R}^n \xrightarrow{\Pi} A$ onto A

PROBLEM 3

Let C be a convex set of dimension r in $\mathbb{R}^n$, b a vector in C .Show that there exists a simplex σ_r of dimension r such that

$$b \in \sigma_r \subseteq C \tag{1}$$

PROBLEM 4

Let A be a set in $\mathbb{R}^n$.Show that

$$cl(A^c) = \left(\text{int}(A)\right)^c \tag{2}$$

PROBLEM 5

The perspective map $\mathbb{R}^n \times \mathbb{R}_{++} \xrightarrow{P} \mathbb{R}^n$ is defined by

$$P(x,t) = \frac{x}{t} \tag{3}$$

Show that

1.if A is a convex set in $\mathbb{R}^n \times \mathbb{R}_{++}$,then $P(A)$ is a convex set in $\mathbb{R}^n$

2.if B is a convex set in $\mathbb{R}^n$,then $P^{-1}(B)$ is a convex set in $\mathbb{R}^{n+1}$

PROBLEM 6

Let A, B be nonempty convex subsets of $\mathbb{R}^n$ .Show that the following are equivalent:

1.A and B are strongly separated, i.e. there exist a nonzero vector $p \in \mathbb{R}^n$,a real number θ and a positive number ε such that

$$\sup_{\alpha \in A}(p\alpha) < \theta - \varepsilon < \theta + \varepsilon < \inf_{\beta \in B}(p\beta)$$

2. $0 \notin \text{closure}(A - B)$

ANSWERS

PROBLEM 1

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be an injective linear map and A a closed set in $\mathbb{R}^n$.

1.Show that $T(A)$ is a closed set.

2.Provide an example of a linear map T and a closed set A such that $T(A)$ is not closed

3.Generalize to injective affine maps

$\boxed{1}$ SHOW T[A] IS CLOSED

WE SPLIT THE PROOF IN SEVERAL PARTS

LET $E = [e_1 \quad e_n]$ BE THE STANDARD BASIS OF R^n

PART 1 $T[E]$ **IS A BASIS OF** $\mathrm{Im}\,T$

PROOF

$T[E]$ IS LINEARLY INDEPENDENT, BECAUSE

$$\sum_{i=1}^{n} \lambda_i T(e_i) = 0 \Rightarrow T\left(\sum \lambda_i e_i\right) = 0 \Rightarrow (T \text{ INJECTIVE})$$

$$\sum \lambda_i e_i = 0 \Rightarrow \lambda_i = 0 \quad \forall i$$

$T[E]$ SPANS $\mathrm{Im}\,T$, BECAUSE

$$y \in \mathrm{Im}\,T \Leftrightarrow y = T(x),\ x \in R^n \Leftrightarrow y = T\left(\sum x_i e_i\right)$$

$$\Leftrightarrow y = \sum x_i T(e_i) \Leftrightarrow y \in \mathrm{SPAN}(T[E])$$

PART 2 THE LINEAR MAP $\mathrm{Im}\,T \xrightarrow{S} R^n$ **DEFINED BY** $S(T(e_i)) = e_i \quad \forall i$ **IS THE INVERSE OF THE LINEAR MAP** $R^n \xrightarrow{T} \mathrm{Im}\,T$

$$\begin{tikzcd}
R^n \arrow[r, "T"] \arrow[dr, "id"'] & \mathrm{Im}\,T \arrow[d, "S"] \arrow[dr, "id"] & \\
& R^n \arrow[r, "T"] & \mathrm{Im}\,T
\end{tikzcd}$$

PROOF: SINCE $T[E]$ IS A BASIS OF $\operatorname{Im} T$, S IS WELL-DEFINED, AND FOR EVERY $x \in \mathbb{R}^n$, $x = \sum x_i e_i$ WE HAVE

$$S(T(x)) = S(T(\sum x_i e_i)) =$$

$$S(\sum x_i T(e_i)) = \sum x_i S(T(e_i)) = \sum x_i e_i = x$$

PART 3: SHOW $T[A]$ IS CLOSED

LET $\omega \xrightarrow{y} T[A]$ BE A SEQUENCE IN $T[A]$ THAT CONVERGES TO SOME POINT $b \in \mathbb{R}^m$. WE WILL SHOW $b \in T[A]$

FOR EACH $\lambda \in \omega$, $y_\lambda = T(\alpha_\lambda)$, $\alpha_\lambda \in A$. THEN

$$b = \lim_{\lambda \to \omega} T(\alpha_\lambda), \quad \alpha_\lambda \in A \quad \forall \lambda \qquad (1)$$

SINCE $T(\alpha_\lambda) \in \operatorname{Im} T$ $\forall \lambda$, (1) IMPLIES

$$b \in CL(\operatorname{Im} T) \qquad (2)$$

$\operatorname{Im} T$ IS A CLOSED SET AS A SUBSPACE, HENCE (2) IMPLIES $b \in \operatorname{Im} T$, I.E.

$$b = T(x) \text{ FOR SOME } x \in \mathbb{R}^n \qquad (3)$$

BY PART 2 WE HAVE

$$x = S(T(x)) = S(b) \qquad (4)$$

AND THEREFORE, BY CONTINUITY OF LINEAR MAPS

$$x \overset{4}{=} S(b) \overset{1}{=} S(\lim_\lambda T(\alpha_\lambda)) =$$

$$\lim_\lambda S(T(\alpha_\lambda)) = \lim_\lambda \alpha_\lambda \quad \text{ie}$$

$$x \in CL(A) \qquad (5)$$

SINCE A IS A CLOSED SET, (5) IMPLIES

$$x \in A \qquad (6)$$

BY (3) (6) $b \in T[A]$, QED

$\boxed{12}$ AN EXAMPLE OF A CLOSED, T(A) NOT CLOSED

LET $A = \{x \in R^2 : x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$ AND $R^2 \xrightarrow{T} R$ $T(x) = x_1$. THEN A IS CLOSED, T IS LINEAR BUT NOT INJECTIVE, AND $T[A] = (0, \infty)$ NOT CLOSED

$\boxed{13}$ LET $R^n \xrightarrow{f} R^m$ INJECTIVE AFFINE MAP AND A A CLOSED SET IN R^n. SHOW THAT $f[A]$ IS CLOSED

PROOF: $f(x) = c + T(x)$ FOR SOME LINEAR T AND T IS INJECTIVE BECAUSE f IS INJECTIVE.

THEN $T[A]$ IS CLOSED. LET $\omega \xrightarrow{S} f[A]$ be A SEQUENCE IN $f[A]$ THAT CONVERGES TO b. WE WILL SHOW $b \in f[A]$, AND THEREFORE THAT $f[A]$ IS CLOSED.

WE HAVE $S_\lambda = f(\alpha_\lambda)$ FOR SOME $\alpha_\lambda \in A$ AND

$$b = \lim_{\lambda \to \infty} f(\alpha_\lambda), \quad \alpha_\lambda \in A \quad \forall_\lambda \qquad (1)$$

BY (1)

$$b = C + \lim_{\lambda \to \infty} T(\alpha_\lambda), \quad \alpha_\lambda \in A \quad \forall_\lambda \qquad \text{ie}$$

$$b - c \in CL(T[A]) = T[A] \qquad \text{ie}$$

$$\exists \ \alpha \in A : \ b - c = T(\alpha) \qquad \text{ie}$$

$$b = c + T(\alpha) = f(\alpha) \qquad \text{ie}$$

$$b \in f[A]$$

PROBLEM 2

Let A be the line in $\mathbb{R}^n$ through the points [1,0,0], [0,1,0]. Derive an explicit formula for the nearest point map $\mathbb{R}^n \xrightarrow{\Pi} A$ onto A

$$\Pi(x) = e_1 + \hat{t} d, \qquad \text{WHERE}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{t} = \frac{1 - x_1 + x_2}{2}$$

PROOF

LET $e_1 = [1 \ 0 \ 0]$, $e_2 = [0 \ 1 \ 0]$ THEN

$\alpha \in A \iff \alpha = (1-t)e_1 + t e_2$ FOR SOME $t \in R \iff$

$\alpha = e_1 + t(e_2 - e_1)$ $t \in R$. HENCE

$$A = e_1 + R \cdot d \qquad (1)$$

$$d = e_2 - e_1 = [-1 \ 1 \ 0] \qquad (2)$$

TAKE ANY $x \in R^3$, AND LET $\hat{\alpha} = \Pi(x)$ BE THE NEAREST POINT TO x IN A. THEN $\hat{\alpha}$ IS THE UNIQUE SOLUTION IN A OF THE VARIATIONAL INEQUALITY

$$(x - \hat{\alpha})(\alpha - \hat{\alpha}) \leq 0 \quad \forall \alpha \in A \qquad (3)$$

$$\hat{\alpha} \in A \qquad (4)$$

BY (1) (2) (3) (4) WE OBTAIN

$$\hat{\alpha} = e_1 + \hat{t} d \quad \text{FOR SOME } \hat{t} \in R \qquad (5)$$

$$(x - e_1 - \hat{t} d)(e_1 + t d - e_1 - \hat{t} d) \leq 0 \quad \forall t \in R \qquad (6)$$

WE WILL COMPUTE $\hat{t}$, TO DETERMINE $\hat{\alpha}$

BY (6)

$$(x - e_1 - \hat{t} d)(t - \hat{t}) d \leq 0 \quad \forall t \in R$$

$$(x - e_1 - \hat{t}d)(t - \hat{t})d \le 0 \quad \forall\, t \in R$$

$$(t - \hat{t})(x - e_1 - \hat{t}d)d \le 0 \quad \forall\, t \in R \qquad (7)$$

BY (7) FOR $t > \hat{t}$

$$(x - e_1 - \hat{t}d)d \le 0 \qquad (8)$$

BY (7) FOR $t < \hat{t}$

$$(x - e_1 - \hat{t}d)d \ge 0 \qquad (8')$$

BY (8) AND (8')

$$(x - e_1 - \hat{t}d)d = 0 \qquad (9)$$

BY (9)

$$(x - e_1)d = \hat{t}\,|d|^2 \quad \text{ie}$$

$$\hat{t} = \frac{(x - e_1)d}{|d|^2} \qquad (10)$$

THEN BY (ε) $d = [-1, 1, 0]$ IS

$$|d|^2 = 2 \qquad (11)$$

$$(x - e_1)d = (x_1 - 1)(-1) + x_2 \cdot 1 + x_3 \cdot 0$$

$$= 1 - x_1 + x_2 \qquad (12)$$

By (11) (12) (10)

$$\hat{t} = \frac{1 - x_1 + x_2}{2} \qquad (13)$$

THEN BY (5) $\hat{a} = e_1 + \hat{t} d$, WITH $\hat{t}$ AS IN (13), QED

PROBLEM 3

Let C be a convex set of dimension r in $\mathbb{R}^n$, b a vector in C. Show that there exists a simplex σ_r of dimension r such that

$$b \in \sigma_r \subseteq C \qquad (1)$$

THE SET $B = \{b\}$ IS AFFINELY INDEPENDENT IN $\mathbb{R}^n$, BECAUSE THE SET

$\left\{ \begin{pmatrix} b \\ 1 \end{pmatrix} \right\}$ IS LINEARLY INDEPENDENT IN $\mathbb{R}^{n+1}$

SINCE $b \in C$, B IS AN AFFINELY INDEPENDENT SUBSET OF C.

BY THE FOLLOWING THEOREM

Theorem 1.3.9 *Let B be an affinely independent subset of a set A in $\mathbb{R}^n$. Then there exists an affine basis for* aff *A that lies in A and contains B.* WEBSTER

THERE EXISTS AN AFFINE BASIS $\hat{B}$ OF AFF(C) SUCH THAT $B \subseteq \hat{B} \subseteq C$ IE

$$b \in \hat{B} \subseteq C \qquad (1)$$

$\hat{B}$ IS AFFINELY INDEPENDENT (2)

$\hat{B}$ IS AFFINELY INDEPENDENT (2)

$\mathrm{AFF}(\hat{B}) = \mathrm{AFF}(C)$ (2')

$\dim(\hat{B}) = \dim(\mathrm{AFF}(\ )) = r$ (3)

LET $\sigma_r = \mathrm{CONV}(\hat{B})$ (4)

BY (2) AND (4) σ_r IS A SIMPLEX (5)

BY (3) AND (4) $\dim(\sigma_r) = r$ (6)

THEN BY (1)

$b \in \hat{B} \subseteq \mathrm{CONV}(\hat{B}) \subseteq \mathrm{CONV}(C) = C$ ie

$b \in \sigma_r \subseteq C$, QED

PROBLEM 4

Let A be a set in $\mathbb{R}^n$. Show that

$$cl(A^c) = (\mathrm{int}(A))^c \qquad (2)$$

Cl(A)c = int(Ac)

$x \in (\mathrm{INT}\ A)^c \iff x \notin \mathrm{INT}\ A \iff$

$\forall \epsilon > 0,\ B_\epsilon(x) \nsubseteq A \iff$

$\forall \epsilon > 0,\ B_\epsilon(x) \cap A^c \neq \emptyset \iff$

$x \in CL(A^c)$

PROBLEM 5

The perspective map $\mathbb{R}^n \times \mathbb{R}_{++} \xrightarrow{P} \mathbb{R}^n$ is defined by

$$P(x,t) = \frac{x}{t} \tag{3}$$

Show that

1.if A is a convex set in $\mathbb{R}^n \times \mathbb{R}_{++}$,then $P(A)$ is a convex set in $\mathbb{R}^n$

2.if B is a convex set in $\mathbb{R}^n$,then $P^{-1}(B)$ is a convex set in $\mathbb{R}^{n+1}$

$\boxed{1}$ LET A BE A CONVEX SET IN $R^n \times R_{++}$. THEN EACH $\alpha \in A$ IS A VECTOR $\alpha = (x, t)$, $x \in R^n$, $t > 0$ AND THEREFORE

$$P(\alpha) = P(x,t) = \frac{x}{t}$$

ie

$$P[A] = \left\{ \frac{x}{t} : (x,t) \in A \right\} \quad (1)$$

TO SHOW THAT $P[A]$ IS CONVEX, TAKE ANY TWO POINTS y, y' IN $P[A]$ AND ANY $\theta \in (0,1)$, AND SHOW THAT $\theta y + (1-\theta)y' \in P[A]$ (?)

BY (1) WE HAVE

$$y = \frac{x}{t}, \quad (x,t) \in A \quad (2)$$

$$y' = \frac{x'}{t'}, \quad (x',t') \in A \quad (2')$$

BY (1) $\theta y + (1-\theta)y' \in P[A]$ iff THERE

EXISTS $(x'', t'') \in A$ SUCH THAT

$$\theta y + (1-\theta) y' = \frac{x''}{t''}$$

HENCE IT SUFFICES TO SHOW

$$\exists\, (x'', t'') \in A$$

$$x'' = t''\theta y + t''(1-\theta) y' \qquad (??)$$

SINCE $(x, t), (x', t') \in A$ AND A IS CONVEX

$$\gamma \begin{pmatrix} x \\ t \end{pmatrix} + (1-\gamma) \begin{pmatrix} x' \\ t' \end{pmatrix} \in A \quad \forall\, \gamma \in (0,1) \quad \text{i.e.}$$

$$\begin{pmatrix} \gamma x + (1-\gamma) x' \\ \gamma t + (1-\gamma) t' \end{pmatrix} \in A, \quad 0 < \gamma < 1 \qquad (3)$$

BY (3) AND (??) IT SUFFICES TO SHOW

THAT THERE EXISTS $\gamma \in (0,1)$ SUCH THAT

$$\boxed{\begin{aligned} \begin{pmatrix} x'' \\ t'' \end{pmatrix} &= \begin{pmatrix} \gamma x + (1-\gamma) x' \\ \gamma t + (1-\gamma) t' \end{pmatrix} \\ x'' &= t''\theta y + t''(1-\theta) y' \end{aligned}}$$

OR EQUIVALENTLY THAT THERE EXISTS

$\gamma \in (0,1)$ THAT SOLVES THE FOLLOWING SYSTEM

$$\boxed{\begin{aligned} \gamma x + (1-\gamma) x' &= t''\theta y + t''(1-\theta) y' \\ \gamma t + (1-\gamma) t' &= t'' \end{aligned}} \qquad (S)$$

$$\gamma t + (1-\gamma)\, t = t'$$

BY (E) (2') THE SYSTEM BECOMES

$$\boxed{\begin{aligned} \gamma x + (1-\gamma) x' &= \frac{t''\theta}{t} x + \frac{t''(1-\theta)}{t'} x' \\ \gamma t + (1-\gamma) t' &= t'' \end{aligned}} \qquad (5)$$

WE TRY THE OBVIOUS, I.E

$$\gamma = \frac{t''}{t}\theta, \quad 1-\gamma = \frac{t''(1-\theta)}{t'}, \text{ i.e.}$$

$$\frac{t''}{t}\theta + \frac{t''(1-\theta)}{t'} = 1$$

$$t''\left(\frac{\theta}{t} + \frac{1-\theta}{t'}\right) = 1$$

$$t'' = \frac{1}{\frac{\theta}{t} + \frac{1-\theta}{t'}} = \frac{t t'}{\theta t' + (1-\theta) t}$$

$$t'' = \frac{t t'}{\theta t' + (1-\theta) t} \qquad (4)$$

$$\gamma = \frac{\theta t'}{\theta t' + (1-\theta) t} \qquad (5)$$

By (5) THERE IS A SOLUTION $\gamma \in (0,1)$

BY (5) THERE IS A SOLUTION $y \in (0,1)$
OF THE SYSTEM (S), QED

$\boxed{2}$

LET B CONVEX SET IN R^n.

SHOW THAT $P^{-1}[B]$ CONVEX SET IN $R^n \times R_{++}$

FIRST WE DESCRIBE $P^{-1}[B]$

$$\begin{pmatrix} x \\ t \end{pmatrix} \in P^{-1}[B] \iff$$

$$P(x,t) \in B, \ t>0 \iff$$

$$\frac{x}{t} \in B, \ t>0 \iff$$

$$x \in tB, \ t>0 \quad \text{ie}$$

$$P^{-1}[B] = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in R^n \times R_{++} : x \in tB \right\} \quad (1)$$

LET $\begin{pmatrix} x \\ t \end{pmatrix} \in P^{-1}[B]$, $\begin{pmatrix} x' \\ t' \end{pmatrix} \in P^{-1}[B]$, $0 < \theta < 1$

SHOW THAT $\theta \begin{pmatrix} x \\ t \end{pmatrix} + (1-\theta) \begin{pmatrix} x' \\ t' \end{pmatrix} \in P^{-1}[B]$

BY (1) THERE EXIST $b, b' \in B$ SUCH THAT

$$x = tb, \quad x' = t'b', \ t>0, \ t'>0 \quad (2)$$

BY (1) AGAIN, IT SUFFICES TO SHOW THAT THERE EXISTS $b'' \in B$ SUCH THAT

EXISTS $b'' \in B$ SUCH THAT

$$\theta x + (1-\theta)x' = (\theta t + (1-\theta)t')b'' \quad (?)$$

BY (2) AND (?) IT SUFFICES TO SHOW

$$\theta t b + (1-\theta) t' b' = (\theta t + (1-\theta)t') b'', \quad b'' \in B$$

ie THAT

$$b'' = \frac{\theta t}{\theta t + (1-\theta)t'} b + \frac{(1-\theta)t'}{\theta t + (1-\theta)t'} b' \in B$$

WHICH IS TRUE BY THE CONVEXITY OF B

PROBLEM 6

Let A, B be nonempty convex subsets of $\mathbb{R}^n$ .Show that the following are equivalent:

1.A and B are strongly separated, i.e. there exist a nonzero vector $p \in \mathbb{R}^n$,a real number θ and a positive number ε such that

$$\sup_{\alpha \in A}(p\alpha) < \theta - \varepsilon < \theta + \varepsilon < \inf_{\beta \in B}(p\beta)$$

2. $0 \notin \text{closure}(A - B)$

Theorem 3.12. *Let C and D be nonempty convex subsets of $\mathbb{R}^n$. Then C and D can be strongly separated if and only if $0 \notin \text{cl}(C - D)$.* ROBINSON

PROOF

$\boxed{1} \rightarrow \boxed{2}$

$\exists\ p \in \mathbb{R}^n,\ \theta, \epsilon \text{ IN } \mathbb{R},\ \epsilon > 0,\ |p| = 1$ st

$$p\alpha < \theta - \epsilon < \theta + \epsilon < p\beta \quad \forall \alpha \in A, \beta \in B \quad (1)$$

By (1) $\forall \alpha \in A$ $\beta \in B$

By (1) $\forall \alpha \in A \quad \beta \in B$

$$p\beta - p\alpha > \theta + \epsilon - (\theta - \epsilon) = 2\epsilon$$

$$p(\beta - \alpha) > 2\epsilon \qquad (2)$$

BY CAUCHY SCHWARZ

$$|p(\beta - \alpha)| \leq |p|\,|\beta - \alpha| = |\beta - \alpha| \qquad (3)$$

By (2) (3)

$$2\epsilon \overset{2}{<} p(\beta - \alpha) \leq |p(\beta-\alpha)| \overset{3}{\leq} |\beta - \alpha| \quad \text{ie}$$

$$\alpha \in A,\ \beta \in B \Rightarrow |\alpha - \beta| \geq 2\epsilon \qquad (4)$$

IF $0 \in CL(A-B)$ THEN $\forall \delta > 0$

$$B_\delta(0) \cap (A - B) \neq \emptyset \qquad (5)$$

BY (5) FOR $\delta = \epsilon$

$\exists\ \alpha, \beta$ in A, B: $|\alpha - \beta| < \epsilon$, CONTRADICTING (4)

$\boxed{2} \rightarrow \boxed{1}$

$$0 \notin CL(A - B) \qquad (1)$$

$A - B$ IS NONEMPTY CONVEX SET BY Linear combinations preserve convexity

$CL(A-B)$ IS CONVEX BY Closure preserves convexity

HENCE $CL(A-B)$ IS NONEMPTY CLOSED CONVEX

Strong separation (SCROLL DOWN TO SEE IT, AS IT WAS

DONE IN CLASS) THEN IMPLIES

$\exists\ p \in \mathbb{R}^n, |p| = 1$ AND $\theta \in \mathbb{R}, \epsilon \in \mathbb{R}$ SUCH THAT

$$px < \theta - \epsilon < \theta + \epsilon < p0 = 0, \quad \forall x \in cl(A - B)$$

AND THEREFORE

$$px < \theta - \epsilon < 0 \quad \forall x \in cl(A - B) \quad (2)$$

FROM (2) WE OBTAIN $\theta < 0$ AND

$$p\alpha < \theta - \epsilon + p\beta \quad \forall \alpha \in A, \beta \in B$$

AND THEREFORE

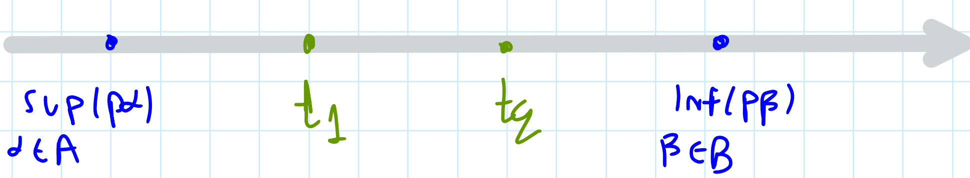
$$\sup_{\alpha \in A}(p\alpha) \le \theta - \epsilon + p\beta \quad \forall \beta \in B$$

AND THEREFORE

$$\sup_{\alpha \in A}(p\alpha) \le \theta - \epsilon + \inf_{\beta \in B}(p\beta) < \inf_{\beta \in B}(p\beta)$$

IE

$$\sup_{\alpha \in A}(p\alpha) < \inf_{\beta \in B}(p \cdot \beta) \quad (3)$$



CHOOSE ANY TWO NUMBERS t_1, t_2 TO SATISFY

$$\sup_{\alpha \in A}(p\alpha) < t_1 < t_2 < \inf_{\beta \in B}(p\beta) \quad (3')$$

AND THEN SOLVE THE EQUATIONS

$$\theta - \epsilon = t_1 \qquad (4)$$

$$\theta + \epsilon = t_2 \qquad (5)$$

TO OBTAIN ϵ

$$\epsilon = \frac{t_2 - t_1}{2} > 0 \qquad (6)$$

$$\theta = \frac{t_2 + t_1}{2} \qquad (7)$$

THEN BY (3') (4) (5)

$$\sup_{\alpha \in A}(p\alpha) < \theta - \epsilon < \theta + \epsilon < \inf_{\beta \in B}(p\beta)$$

Separation of a closed convex set from a point outside the set

2.4.4 Theorem

Buzasau - Shetty

Let S be a nonempty closed convex set in R^n and $\mathbf{y} \notin S$. Then there exists a nonzero vector $\mathbf{p}$ and a scalar α such that $\mathbf{p}'\mathbf{y} > \alpha$ and $\mathbf{p}'\mathbf{x} \leq \alpha$ for each $\mathbf{x} \in S$.

THEOREM 29.1 * (*a separation theorem*). *Let a non-empty closed convex set X and a point a not contained in it, i.e. $a \notin X$, be given in R^n. Then, there exists a hyperplane that includes a in one of its half-spaces and X in the other of the half-spaces.*

NIKAIDO

* The proof is due to von Neumann and Morgenstern [24].

Corollary 2.4.7 *In $\mathbb{R}^n$ let A be a closed convex set and let $\mathbf{b}$ be a point not lying in A. Then A and $\{\mathbf{b}\}$ can be strictly separated by a hyperplane in $\mathbb{R}^n$.* $\square$

WEBSTER

Proposition 3.5. *Let C be a nonempty closed convex subset of $\mathbb{R}^n$, and let $x \notin C$. Then $\{x\}$ and C can be strongly separated.*

ROBINSON

A = NONEMPTY, CLOSED CONVEX SET IN R^n, $x \in R^n$, $x \notin A$

$R^n \xrightarrow{\pi} A$ NEAREST POINT MAP ONTO A,

$$p = \frac{x - \pi(x)}{|x - \pi(x)|}, \quad p\,\pi(x) \le \theta \le px$$

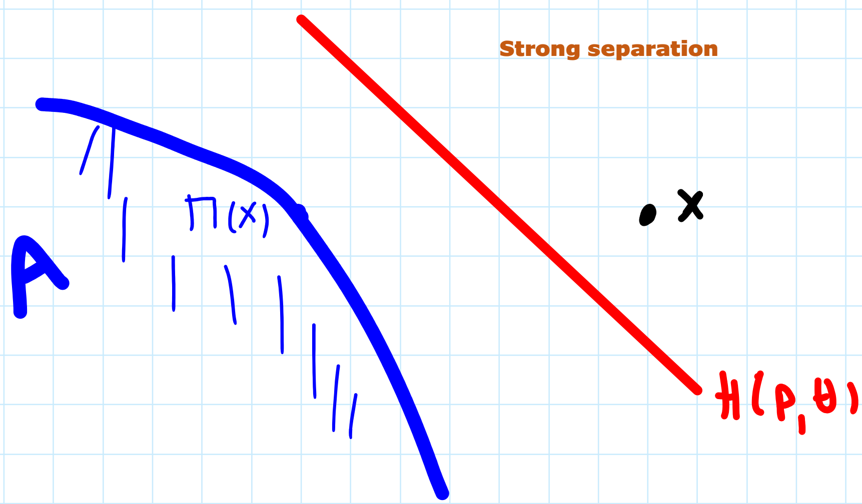
$H(p, \theta) = \{x \in R^n : px = \theta\}$ THEN

$\boxed{1}$ IF $p \cdot \pi(x) < \theta < p \cdot x$ THEN $H(p, \theta)$ STRONGLY SEPARATES A FROM x ie

$A \subseteq H_{--}(p, \theta)$ $\quad \sup_{\alpha \in A}(p\alpha) < \theta$

$x \in H_{++}(p, \theta)$ $\quad \theta < px$

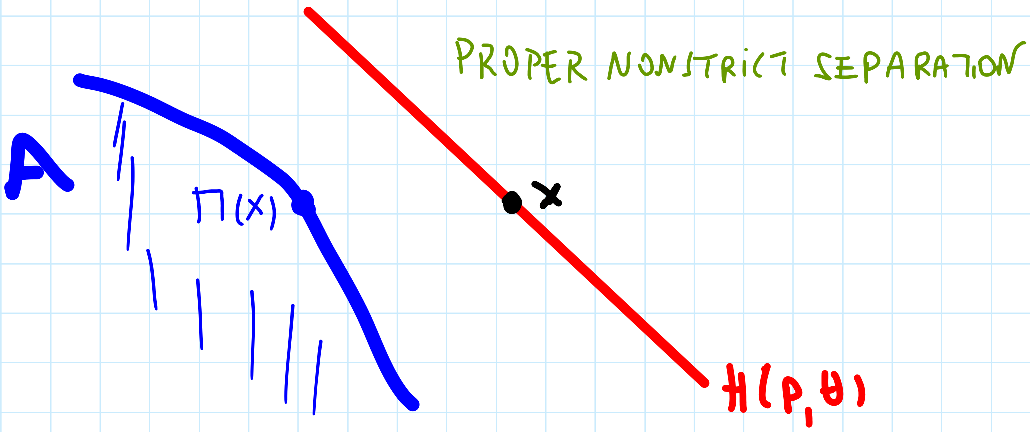
Strong separation



$\boxed{2}$ IF $\theta = px$ THEN $H(p, \theta)$ PROPERLY SEPARATES A FROM x

$A \subseteq H_{--}(p,\theta)$ $\quad \sup_{\alpha \in A}(p \cdot \alpha) < \theta$

$x \in H(p,\theta)$ $\quad \theta = px$



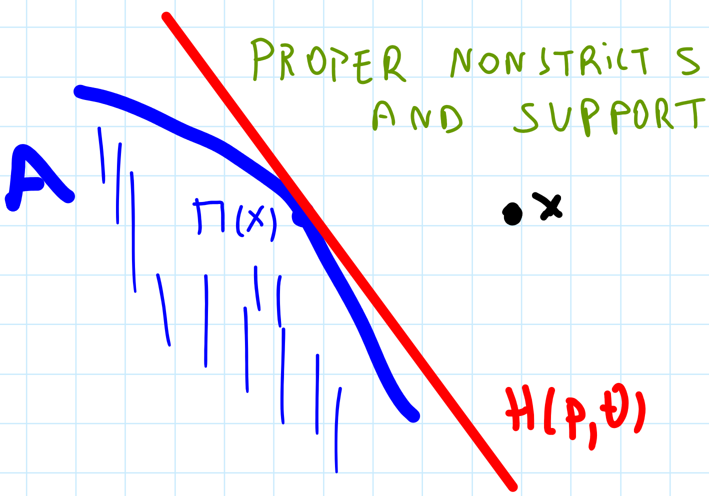
$\boxed{3}$ IF $\theta = p \cdot \pi(x)$ THEN

- $H(p,\theta)$ SUPPORTS A AT $\pi(x)$, i.e.

  $\pi(x) \in A \cap H(p,\theta)$, $A \subseteq H_{-}(p,\theta)$

- $H(p,\theta)$ PROPERLY SEPARATES A FROM b

  $A \subseteq H_{-}(p,\theta)$, $x \in H_{++}(p,\theta)$



PROOF

Hahn banach theorem,geometric form SHOWS $\exists\, p \in \mathbb{R}^n$, $|p| = 1$:

Hahn banach theorem,geometric form SHOWS $\exists\, p \in \mathbb{R}^n$, $|p| = 1$:

$$\sup_{\alpha \in A} (p \cdot \alpha) = p \cdot \pi(x) < px \quad (1)$$

HENCE $\exists\, \theta \in \mathbb{R}$: $p\pi(x) \leq \theta \leq px$

AND FOR $p\pi(x) < \theta < px$

$$\sup_{\alpha \in A} (p \cdot \alpha) = p \cdot \pi(x) < \theta < px$$

AND THEFORE

$$A \subseteq H_{--}(p,\theta), \quad x \in H_{++}(p,\theta)$$

FOR $\theta = px$

$$A \subseteq H_{--}(p,\theta), \quad x \in H(p,\theta)$$

FOR $\theta = p \cdot \pi(x)$

$$\pi(x) \in \partial A \cap H(p,\theta)$$

$$A \subseteq H_{-}(p,\theta), \quad x \in H_{++}(p,\theta)$$