Inclassexam2023july,answers

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inclassexam2023july

In class exam July 7, 2023

please answer all questions

PROBLEM 1

Let $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ be a continuous function and let A be a bounded set in $\mathbb{R}^n$ .Show that

$$f(\text{closure}(A)) = \text{closure}(f(A))$$

PROBLEM 2

Let $A_{m\times n}, A'_{m'\times n}$ be real matrices. Let $b \in \mathbb{R}^m, b' \in \mathbb{R}^{m'}$ .Show that exactly one of the following statements is true (either the primal or the dual, but never both)

primal
there exists $x \in \mathbb{R}^n$ such that $Ax = b, A'x \le b', x \ge 0$

dual
there exist $p \in \mathbb{R}^m, p' \in \mathbb{R}^{m'}$ such that $pA + p'A' \ge 0, pb + p'b' < 0, p' \ge 0$

Hint: use slack variables to reduce the primal of this problem to the primal of Farkas' lemma.

The notation $x \ge 0$ means that each component x_i of the vector x is greater than or equal to zero.

$pA, p'A'$ denote the vector-matrix products $p_{1\times m}A_{m\times n}, p'_{1\times m'}A'_{m'\times n}$

$pb, p'b'$ denote the dot products $p_{1\times m}b_{m\times 1}, p'_{1\times m'}b'_{m'\times 1}$

ANSWERS TO PROBLEM 1

SHOW $f[CL\,A] \subseteq CL\,f[A]$

LET $y \in f[\text{CL } A]$. THEN $y = f(x)$ FOR SOME $x \in \text{CL } A$. BY THE SEQUENTIAL CHARACTERIZATION OF CLOSURE $x = \lim_{\lambda \to \infty} \alpha_\lambda$ FOR SOME SEQUENCE $\omega \xrightarrow{\alpha} A$.

BY THE CONTINUITY OF f THEN,

$$y = f(x) = f(\lim_{\lambda \to \infty} \alpha_\lambda) = \lim_{\lambda \to \infty} f(\alpha_\lambda) \qquad (1)$$

SINCE $f(\alpha_\lambda) \in f[A]$ $\forall \lambda$, (1) IMPLIES THAT $y \in \text{CL } f[A]$, QED

SHOW $\text{CL } f[A] \subseteq f[\text{CL } A]$

LET $y \in \text{CL } f[A]$. THEN THERE IS A SEQUENCE $\omega \xrightarrow{S} f[A]$ SUCH THAT

$$y = \lim_{\lambda \to \omega} S_\lambda \qquad (1)$$

SINCE $S_\lambda \in f[A]$ $\forall \lambda$, $S_\lambda = f(\alpha_\lambda)$ FOR SOME $\alpha_\lambda \in A$, HENCE BY (1)

$$y = \lim_{\lambda \to \infty} f(\alpha_\lambda) \qquad (2)$$

SINCE A IS A BOUNDED SET AND $\alpha_\lambda \in A$ FOR ALL λ, THE SEQUENCE $\omega \xrightarrow{\alpha} A$ IS BOUNDED BY THE BOLZANO-WEIERSTRASS THEOREM THE

BY THE BOLZANO-WEIERSTRASS THEOREM THE BOUNDED SEQUENCE $\omega \xrightarrow{\alpha} A$ HAS A CONVERGENT SUBSEQUENCE α'

$$\begin{tikzcd} \omega \arrow[r, "\alpha"] & A \\ \omega \arrow[u, "\nu", "\lambda"'] \arrow[ur, "\alpha'"] & \end{tikzcd} \qquad \lim_{\lambda\to\infty} \alpha'_\lambda = \lim_{\lambda\to\infty} \alpha_{\nu_\lambda} = b \qquad (3)$$

SINCE $\alpha'_\lambda \in A \ \forall \lambda \in A$, (3) IMPLIES

$$b \in \mathrm{CL}\, A \qquad (4)$$

BY (3) WE HAVE

$$\begin{tikzcd} \omega \arrow[r, "\alpha"] & A \subseteq R^n \arrow[r, "f"] & R^m \\ \omega \arrow[u, "\nu", "\lambda"'] \arrow[ur, "\alpha'"] & & \end{tikzcd} \qquad (5)$$

ie $f \circ \alpha'$ IS A SUBSEQUENCE OF $f \circ \alpha$ AND THEREFORE

$$y \overset{2}{=} \lim_{\lambda\to\infty} f(\alpha_\lambda) = \lim_{\lambda\to\infty} f(\alpha'_\lambda) \qquad (6)$$

THEN BY THE CONTINUITY OF f

$$y \overset{6}{=} \lim_{\lambda\to\infty} f(\alpha'_\lambda) = f\left(\lim_{\lambda\to\infty} \alpha'_\lambda\right) \overset{3}{=} f(b) \qquad \text{ie}$$

$$y = f(b) \qquad (7)$$

THEN BY (4) AND (7), $y \in f[\mathrm{CL}\, A]$, QED

ANSWERS TO PROBLEM 2

LET $s = b' - A'x \geqslant 0$

THEN THE PRIMAL BECOMES

$$\exists\, x \in R^n,\ s \in R^{m'} \text{ SUCH THAT}$$

$$Ax = b,\quad A'x + s = b',\quad x \geqslant 0,\quad s \geqslant 0$$

LET
$$M = \begin{array}{cc} & \begin{array}{cc} n & m' \end{array} \\ \begin{array}{c} m \\ m' \end{array} & \begin{bmatrix} A & 0 \\ A' & I_{m'} \end{bmatrix} \end{array}, \quad z = \begin{bmatrix} x \\ s \end{bmatrix} \in R^{n+m'}$$

$$c = \begin{bmatrix} b \\ b' \end{bmatrix} \in R^{m+m'}$$

THEN THE PRIMAL BECOMES

$$\exists\, z \in R^{n+m'} :\ Mz = c,\ z \geqslant 0 \qquad (P)$$

BY FARKAS LEMMA THE DUAL OF (P) IS

$$\exists\, y \in R^{m+m'} :\ yM \geqslant 0,\ yc < 0 \qquad (D)$$

WE WRITE $y = [p \ \ p']$, $p \in R^m$, $p' \in R^{m'}$

THEN

WE WRITE $y = [p \quad p'],\ p \in \mathbb{R}^m,\ p' \in \mathbb{R}^{m'}$

THEN

$$yM = [p \quad p'] \begin{bmatrix} A & 0 \\ A' & I_{m'} \end{bmatrix} = [pA + p'A',\ p']$$

$$yc = [p \quad p'] \begin{bmatrix} b \\ b' \end{bmatrix} = pb + p'b'$$

AND THEREFORE THE DUAL BECOMES

$\exists p \in \mathbb{R}^m, p' \in \mathbb{R}^{m'}$ SUCH THAT

$$pA + p'A' \geq 0, \quad p' \geq 0$$

$$pb + p'b' < 0$$