Marginal Likelihood for a Normal Model (Unknown Mean, Known Variance)

Model

Let $x_1, \dots, x_N$ be i.i.d.

$$x_i \mid \mu \sim \mathcal{N}(\mu, \sigma^2), \qquad \sigma^2 \text{ known},$$

and place the conjugate prior

$$\mu \sim \mathcal{N}(\mu_0, \tau_0^2).$$

Write $\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i$ and $S = \sum_{i=1}^N (x_i - \bar{x})^2$.

Step 1: Likelihood and a decomposition using $\bar{x}$

The likelihood is

$$\begin{aligned} p(\mathbf{x} \mid \mu) &= \prod_{i=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^N (x_i - \mu)^2\right). \end{aligned}$$

Now expand and complete the square:

$$\begin{aligned} \sum_{i=1}^N (x_i - \mu)^2 = \sum_{i=1}^N \left(x_i^2 - 2\mu x_i + \mu^2\right) &= \sum_{i=1}^N x_i^2 - 2\mu \sum_{i=1}^N x_i + N\mu^2 \\ = \sum_{i=1}^N x_i^2 - 2N\bar{x}\,\mu + N\mu^2 &= \sum_{i=1}^N x_i^2 - N\bar{x}^2 + N(\mu - \bar{x})^2. \end{aligned}$$

Using the identity $\sum_{i=1}^N (x_i - \bar{x})^2 = \sum_{i=1}^N x_i^2 - N\bar{x}^2$, we obtain the key decomposition

$$\boxed{\sum_{i=1}^N (x_i - \mu)^2 = S + N(\mu - \bar{x})^2.} \tag{1}$$

Plugging (1) into the likelihood yields

$$p(\mathbf{x} \mid \mu) = (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{S}{2\sigma^2}\right) \exp\left(-\frac{N}{2\sigma^2}(\mu - \bar{x})^2\right). \tag{2}$$

So, as a function of μ, the likelihood is proportional to a Normal kernel centered at $\bar{x}$ with variance σ^2/N.

Step 2: Marginal likelihood (prior predictive)

The marginal likelihood is

$$p(\mathbf{x}) = \int p(\mathbf{x} \mid \mu)\, p(\mu)\, d\mu.$$

Substitute (2) and the prior density

$$p(\mu) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\Big(-\frac{(\mu-\mu_0)^2}{2\tau_0^2}\Big):$$

$$p(\mathbf{x}) = (2\pi\sigma^2)^{-N/2} \exp\Big(-\frac{S}{2\sigma^2}\Big) \int \exp\Big(-\frac{N}{2\sigma^2}(\mu-\bar{x})^2\Big) \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\Big(-\frac{(\mu-\mu_0)^2}{2\tau_0^2}\Big)\, d\mu.$$

Combine the exponents. Define the "likelihood variance" $v = \sigma^2/N$. Then

$$\frac{N}{\sigma^2}(\mu-\bar{x})^2 + \frac{1}{\tau_0^2}(\mu-\mu_0)^2 = \Big(\frac{1}{v}+\frac{1}{\tau_0^2}\Big)\mu^2 - 2\Big(\frac{\bar{x}}{v}+\frac{\mu_0}{\tau_0^2}\Big)\mu + \Big(\frac{\bar{x}^2}{v}+\frac{\mu_0^2}{\tau_0^2}\Big).$$

Let

$$A = \frac{1}{v}+\frac{1}{\tau_0^2}, \qquad m = \frac{\bar{x}/v + \mu_0/\tau_0^2}{A}, \qquad V = A^{-1} = \Big(\frac{1}{v}+\frac{1}{\tau_0^2}\Big)^{-1}.$$

Then completing the square gives

$$\frac{N}{\sigma^2}(\mu-\bar{x})^2 + \frac{1}{\tau_0^2}(\mu-\mu_0)^2 = A(\mu-m)^2 + \underbrace{\Big(\frac{\bar{x}^2}{v}+\frac{\mu_0^2}{\tau_0^2} - Am^2\Big)}_{\text{does not depend on } \mu}.$$

Hence

$$\begin{aligned}\int \exp\Big(-\frac{1}{2}\Big[\frac{N}{\sigma^2}(\mu-\bar{x})^2 + \frac{1}{\tau_0^2}(\mu-\mu_0)^2\Big]\Big)\, d\mu &= \exp\Big(-\frac{1}{2}\Big[\frac{\bar{x}^2}{v}+\frac{\mu_0^2}{\tau_0^2} - Am^2\Big]\Big) \int \exp\Big(-\frac{A}{2}(\mu-m)^2\Big)\, d\mu \\ &= \exp\Big(-\frac{1}{2}\Big[\frac{\bar{x}^2}{v}+\frac{\mu_0^2}{\tau_0^2} - Am^2\Big]\Big)\sqrt{\frac{2\pi}{A}}.\end{aligned}$$

Accounting for the prefactor $1/\sqrt{2\pi\tau_0^2}$, the integral becomes

$$\frac{1}{\sqrt{2\pi\tau_0^2}}\sqrt{\frac{2\pi}{A}} \exp\Big(-\frac{1}{2}\Big[\frac{\bar{x}^2}{v}+\frac{\mu_0^2}{\tau_0^2} - Am^2\Big]\Big) = \sqrt{\frac{V}{\tau_0^2}} \exp\Big(-\frac{(\bar{x}-\mu_0)^2}{2(\tau_0^2+v)}\Big),$$

where in the last equality one can simplify algebraically using $V^{-1} = v^{-1} + \tau_0^{-2}$. Noting that

$$\sqrt{\frac{V}{\tau_0^2}} = \frac{1}{\sqrt{\tau_0^2+v}}\sqrt{v} \quad \text{and} \quad \frac{1}{\sqrt{2\pi(\tau_0^2+v)}} \exp\Big(-\frac{(\bar{x}-\mu_0)^2}{2(\tau_0^2+v)}\Big) = \mathcal{N}(\bar{x} \mid \mu_0, \tau_0^2+v),$$

we obtain the clean closed form below.

Final result

With $v = \sigma^2/N$,

$$\boxed{p(\mathbf{x}) = (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{S}{2\sigma^2}\right) \mathcal{N}\left(\bar{x} \mid \mu_0, \tau_0^2 + \frac{\sigma^2}{N}\right).} \tag{3}$$

Equivalently, written out:

$$p(\mathbf{x}) = (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{S}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\left(\tau_0^2 + \sigma^2/N\right)}} \exp\left(-\frac{(\bar{x} - \mu_0)^2}{2\left(\tau_0^2 + \sigma^2/N\right)}\right).$$

A convenient log form is

$$\log p(\mathbf{x}) = -\frac{N}{2} \log(2\pi\sigma^2) - \frac{S}{2\sigma^2} - \frac{1}{2} \log\left(2\pi\left(\tau_0^2 + \frac{\sigma^2}{N}\right)\right) - \frac{(\bar{x} - \mu_0)^2}{2\left(\tau_0^2 + \sigma^2/N\right)}.$$