Statistics for Business

Random Variables and Probability Distributions, Special Discrete and Continuous Probability Distributions

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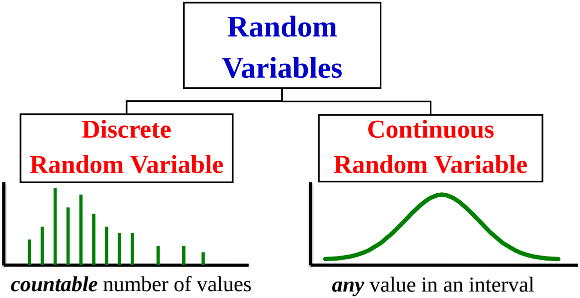
Random Variables – I

Basics

Definition

A ***random variable*** X is a a function or rule that assigns a **number** to each outcome of an experiment.

Think of this as the numerical summary of a random outcome.



Random Variables – II

Basics

Examples

- X = GPA for a randomly selected student
- X = number of contracts a shipping company has pending at a randomly selected month of the year
- X = number on the upper face of a randomly tossed die
- X = the price of crude oil during a randomly selected month.

Discrete Random Variables

- A ***discrete random variable*** can only take on a countable number of values

Examples

- Roll a die twice. Let X be the number of times 4 comes up:
  - then X could be 0, 1, or 2 times
- Toss a coin 5 times. Let X be the number of heads:
  - then $X = 0, 1, 2, 3, 4$, or 5

Discrete Probability Distributions – I

- The ***probability distribution*** for a ***discrete random variable*** X resembles the relative frequency distributions. It is a graph, table or formula that gives the possible values of X and the probability $P(X = x)$ associated with each value.
- This must satisfy
  1. $0 \leq P(x) \leq 1$, for all x.
  2. $\sum_{\text{all } x} P(x) = 1$, the individual probabilities sum to 1.
- The ***cumulative probability function***, denoted by $F(x_0)$, shows the probability that X is less than or equal to a particular value, x_0 :

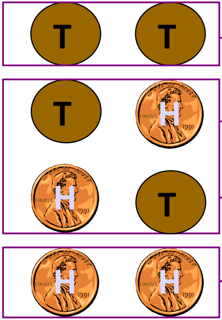
$$F(x_0) = \Pr(X \leq x_0) = \sum_{x \leq x_0} P(x)$$

Discrete Probability Distributions – II

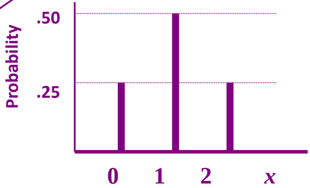
- **Random Experiment**: Toss 2 Coins. Let (the random variable) $X = \#$ heads.

Discrete Probability Distributions – III

4 possible outcomes **Probability Distribution**



x Value	Probability	Cum. Prob.
0	1/4 = .25	1/4 = .25
1	2/4 = .50	3/4 = .75
2	1/4 = .25	4/4 = 1.00



- **Random Experiment**: Let the random variable S be the number of days it will snow in the last week of January

Discrete Probability Distributions – IV

(cumulative) Probability distribution of S								
Outcome	0	1	2	3	4	5	6	7
Probability	0.20	0.25	0.20	0.15	0.10	0.05	0.04	0.01
CDF	0.20	0.45	0.65	0.80	0.90	0.95	0.99	1.00

Moments of Discrete Prob. Distributions – I

- ***Expected Value*** (or ***mean***) of a discrete distribution (*weighted average*)

$$\mu_X = \mathrm{E}(X) = \sum_{\text{all } x} x \cdot P(x).$$

- ***Variance*** of a discrete random variable X (*weighted average...*)

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}\left[(X - \mu_X)^2\right] = \sum_{\text{all } x} (x - \mu_X)^2 \cdot P(x)$$

- ***Standard Deviation*** of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_{\text{all } x} (x - \mu)^2 P(x)}$$

Moments of Discrete Prob. Distributions – II

Example

Consider the experiment of tossing 2 coins, and $X = \#$ of heads. Then

$$\begin{aligned} \mu &= \mathrm{E}(X) = \sum_x xP(x) \\ &= (0 \times 0.25) + (1 \times 0.50) + (2 \times 0.25) = 1 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{\sum_x (x-\mu)^2 P(x)} \\ &= \sqrt{(0-1)^2(.25) + (1-1)^2(.50) + (2-1)^2(.25)} \\ &= \sqrt{.50} = 0.707 \end{aligned}$$

Moments of Discrete Prob. Distributions – III

Example (Number of days it will snow in January)

$\mu_S = \mathrm{E}(S) = \sum_s s \cdot P(s) =$
$= 0{\cdot}0.2+1{\cdot}0.25+2{\cdot}0.2+3{\cdot}0.15+4{\cdot}0.1+5{\cdot}0.05+6{\cdot}0.04+7{\cdot}0.01 = 2.06$

$\sigma_S^2 = \mathrm{Var}(S) = \sum_s (s - \mathrm{E}(S))^2 \cdot P(s) =$
$= (0-2.06)^2{\cdot}0.2+(1-2.06)^2{\cdot}0.25+(2-2.06)^2{\cdot}0.2+(3-2.06)^2{\cdot}0.15$
$+(4-2.06)^2 \cdot 0.1 + (5-2.06)^2 \cdot 0.05 + (6-2.06)^2 \cdot 0.04$
$+(7-2.06)^2 \cdot 0.01 = 2.94$

Remark (Rules for Moments)

Let a and b be any constants and let $Y = a + bX$. Then

$$\begin{aligned} \mathrm{E}[a+bX] &= a + b\mathrm{E}[X] = a + b\mu_x \\ \mathrm{Var}[a+bX] &= b^2\mathrm{Var}[X] = b^2\sigma_x^2 \Rightarrow \sigma_Y = |b|\sigma_x \end{aligned}$$

- The above imply that $\mathrm{E}[a] = a$ and $\mathrm{Var}[a] = 0$

Prob. Density and Distribution Function – I

- The ***probability density function*** (or *pdf*), $f(x)$, of continuous random variable X has the following properties

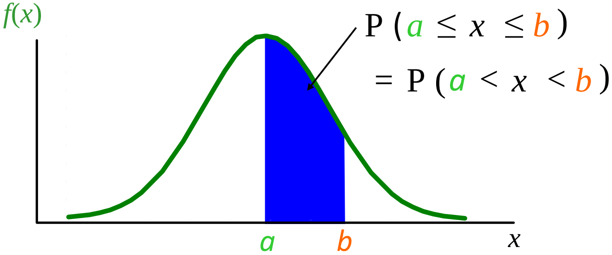
1. $f(x) > 0$ for all values of x (x takes a range of values, $\mathbb{R}_X$).
2. The area under the probability density function $f(x)$ over all values of the random variable X is equal to 1 (recall that $\sum_{\text{all } x} P(x) = 1$ for discrete r.v.)

$$\int_{\mathbb{R}_X} f(x)dx = 1.$$

Prob. Density and Distribution Function – II

3. The probability that X lies between two values is the area under the density function graph between the two values:

$$\Pr(a \leq X \leq b) = \Pr(a < X < b) = \int_a^b f(x)dx$$



Note that the probability of any individual value is zero

Prob. Density and Distribution Function – III

4. The ***cumulative density function*** (or ***distribution function***) $F(x_0)$, which expresses the probability that X does not exceed the value of x_0, is the area under the probability density function $f(x)$ from the minimum x value up to x_0

$$F(x_0) = \int_{x_{\min}}^{x_0} f(x)dx.$$

5. It follows that

$$\Pr(a \leq X \leq b) = \Pr(a < X < b) = F(b) - F(a)$$

Moments of Continuous Distributions – I

- ***Expected Value*** (or ***mean***) of a continuous distribution

$$\mu_X = \mathrm{E}(X) = \int_{\mathbb{R}_X} x f(x) dx.$$

- ***Variance*** of a continuous random variable X

$$\sigma_X^2 = \mathrm{Var}(X) = \int_{\mathbb{R}_X} (x - \mu_X)^2 f(x) dx$$

- ***Standard Deviation*** of a continuous random variable X

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\int_{\mathbb{R}_X} (x - \mu_X)^2 f(x) dx}$$

Moments of Continuous Distributions – II

Remark (Rules for Moments Apply)

Let c and d be any constants and let $Y = c + dX$. Then

$$\begin{aligned}\mathrm{E}[c + dX] &= c + d\mathrm{E}[X] = c + d\mu_x \\ \mathrm{Var}[c + dX] &= d^2\mathrm{Var}[X] = d^2\sigma_x^2 \Rightarrow \sigma_Y = |d|\sigma_x\end{aligned}$$

Remark (**Standardized Random Variable**)

*An **important special case** of the previous results is*

$$Z = \frac{X - \mu_x}{\sigma_x},$$

$$\textit{for which} \quad : \quad \begin{array}{c}\mathrm{E}(Z) = 0 \\ \mathrm{Var}(Z) = 1\end{array}.$$

Bernoulli Distribution

- Consider only two outcomes: "*success*" or "*failure*". Let p denote the probability of success, and $1-p$ be the probability of failure.
- Define random variable X: $x=1$ if success, $x=0$ if failure.
- Then the *Bernoulli probability function* is

$$P(X=0)=(1-p) \text{ and } P(X=1)=p$$

- Moreover:

$$\begin{aligned}
\mu_X &= \mathrm{E}(X)=\sum_{\text{all } x} x \cdot P(x)=0\cdot(1-p)+1\cdot p=p \\
\sigma_X^2 &= \mathrm{Var}(X)=\mathrm{E}[(X-\mu_X)^2]=\sum_{\text{all } x}(x-\mu_X)^2\cdot P(x) \\
&= (0-p)^2(1-p)+(1-p)^2 p=p(1-p)
\end{aligned}$$

Binomial Distribution – I

- A fixed number of observations, n
  - e.g., 15 tosses of a coin; ten light bulbs taken from a warehouse
- Two mutually exclusive and collectively exhaustive categories
  - e.g., head or tail in each toss of a coin; defective or not defective light bulb
  - Generally called "*success*" and "*failure*"
  - Probability of success is p, probability of failure is $1 - p$
- Constant probability for each observation
  - e.g., Probability of getting a tail is the same each time we toss the coin
- Observations are independent
  - The outcome of one observation does not affect the outcome of the other

Binomial Distribution – II

- Examples:
  - A manufacturing plant labels items as either defective or acceptable
  - A firm bidding for contracts will either get a contract or not
  - A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
  - New job applicants either accept the offer or reject it
- To calculate the probability associated with each value we use combinatorics:

$$P(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}; \quad x = 0, 1, 2, \dots, n$$

Binomial Distribution – III

- $P(x)$ = probability of x successes in n trials, with probability of success p on each trial; x = number of 'successes' in sample (nr. of trials n); $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$

Example

What is the probability of one success in five observations if the probability of success is 0.1?

- Here $x = 1, n = 5$, and $p = 0.1$. So

$$\begin{aligned} P(x &= 1) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \frac{5!}{1!(5-1)!} (0.1)^1 (1-0.1)^{5-1} = 5(0.1)(0.9)^4 = 0.32805 \end{aligned}$$

Binomial Distribution

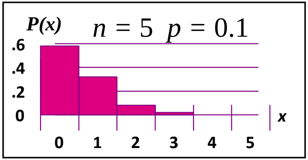
Moments and Shape

$$\mu = E(X) = np$$
$$\sigma^2 = Var(X) = np(1-p) \Rightarrow \sigma = \sqrt{np(1-p)}$$

- The shape of the binomial distr. depends on the values of p and n

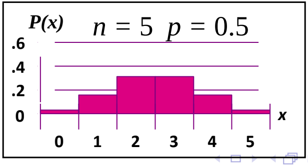
$\mu = np = (5)(0.1) = 0.5$

$\sigma = \sqrt{np(1-p)} = \sqrt{(5)(0.1)(1-0.1)} = 0.6708$



$\mu = np = (5)(0.5) = 2.5$

$\sigma = \sqrt{np(1-p)} = \sqrt{(5)(0.5)(1-0.5)} = 1.118$



Normal Distribution – I

- The *normal distribution* is the most important of all probability distributions. The probability density function of a **normal random variable** is given by

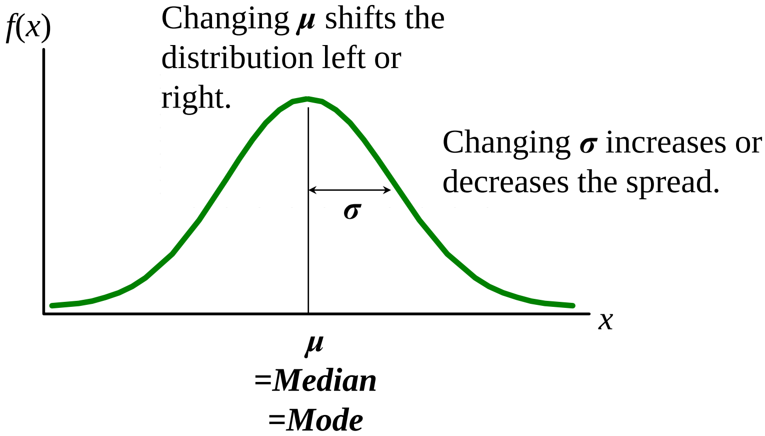
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \ -\infty < x < +\infty,$$

and we usually write $X \sim N(\mu_x, \sigma_x^2)$

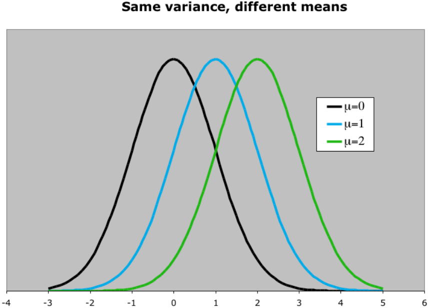
  - The normal distribution closely approximates the probability distributions of a wide range of random variables
  - Distributions of sample means approach a normal distribution given a "large" sample size
  - Computations of probabilities are direct and elegant

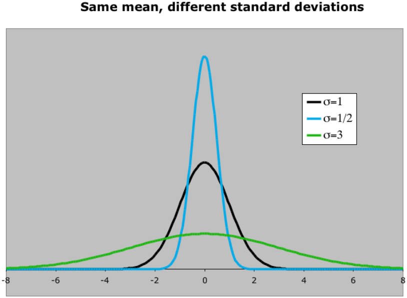
Normal Distribution – II

- The shape and location of the normal curve changes as the mean (μ) and standard deviation (σ) change



Normal Distribution – III





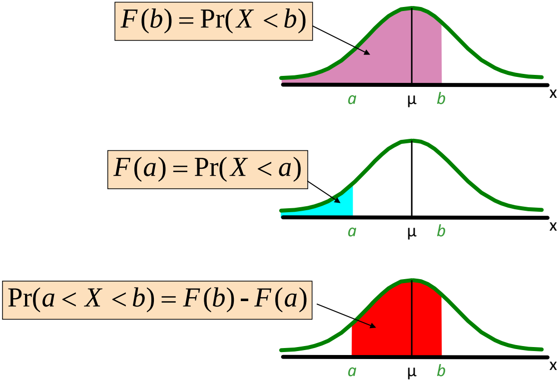
- For a normal random variable X with mean μ and variance σ^2, i.e., $X \sim N(\mu, \sigma^2)$), the cumulative distribution function is

$$F(x_0) = \Pr(X \leq x_0),$$

Normal Distribution – IV

while the probability for a range of values is measured by the area under the curve

$$\Pr(a < X < b) = F(b) - F(a)$$



Normal Distribution – V

- Any normal distribution (with any mean and variance combination) can be transformed into the standardized normal distribution (Z), with mean 0 and variance 1:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

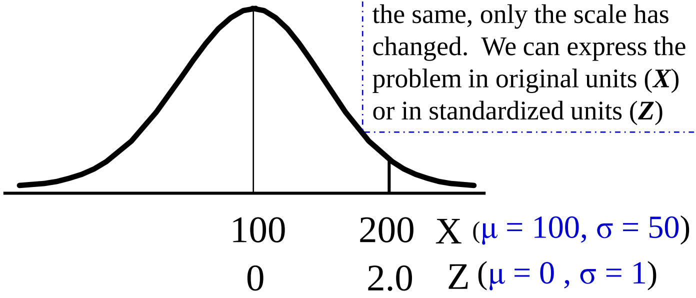
- **Example**: If $X \sim N(100, 50^2)$, the Z value for $X = 200$ is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2$$

This says that $X = 200$ is two standard deviations (2 increments of 50 units) above the mean of 100.

Normal Distribution – VI

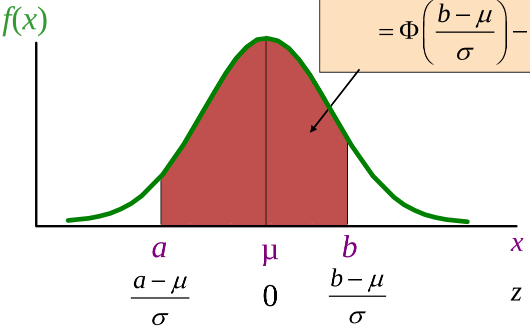
Note that the distribution is the same, only the scale has changed. We can express the problem in original units (***X***) or in standardized units (***Z***)



Finding Normal Probabilities – I

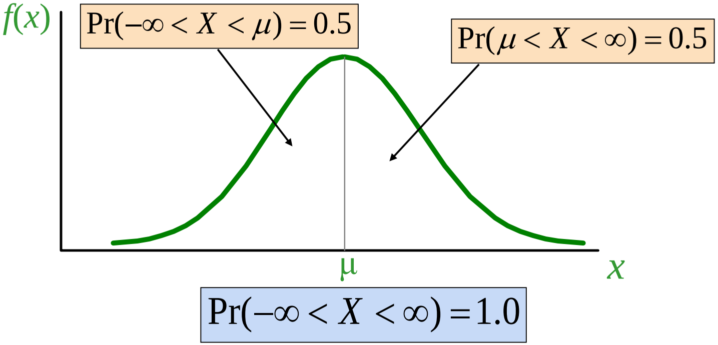
$$\Pr(a < X < b) = \Pr\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$



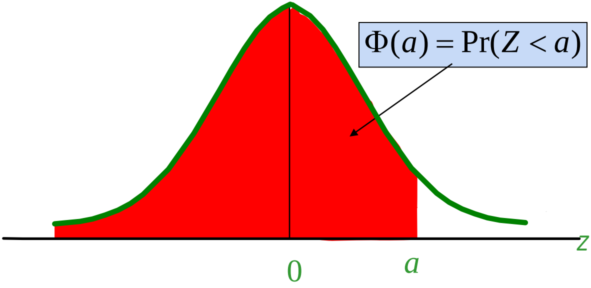
Finding Normal Probabilities – II

- The *total area under the curve is 1.0*, and the curve is symmetric, so half is above the mean, half is below



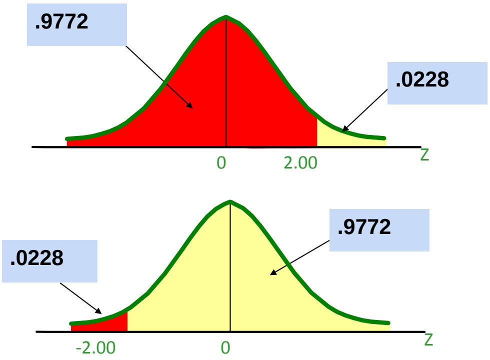
Finding Normal Probabilities – III

- Table with cumulative *standard normal distribution*: For a given Z-value a, the table shows $\Phi(a)$ (the area under the curve from negative infinity to a)



Finding Normal Probabilities – IV

- **Example**: Suppose we are interested in $\Pr(Z < 2)$ – from the previous example. For negative $Z-$values, we use the fact that the distribution is symmetric to find the needed probability (e.g. $\Pr(Z < -2)$).

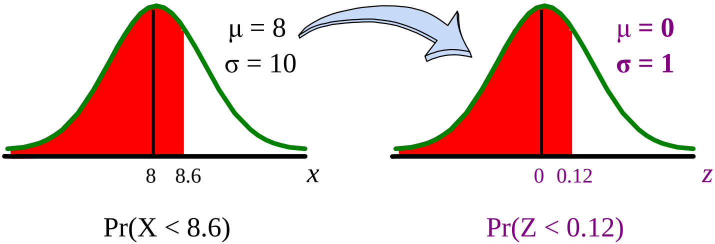


Finding Normal Probabilities – V

- **Example**: Suppose X is normal with mean 8.0 and standard deviation 5.0. Find $\Pr(X < 8.6)$.

$$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12;$$

$$\Phi(0.12) \; = \; 0.5478$$



Finding Normal Probabilities – VI

- **Example (Upper Tail Probabilities)**: Suppose X is normal with mean 8.0 and standard deviation 5.0. Find $\Pr(X > 8.6)$.

$$\begin{aligned}\Pr(X &> 8.6) = \Pr(Z > 0.12) = 1 - \Pr(Z \leq 0.12)\\ &= 1 - 0.5478 = 0.4522\end{aligned}$$

- **Example (Finding X for a Known Probability)** Suppose $X \sim N(8, 5^2)$. Find a X value so that only 20% of all values are below this X.
  1. Find the Z-value for the known probability $\Phi(.84) = .7995$, so a 20% area in the lower tail is consistent with a Z-value of -0.84.

Finding Normal Probabilities – VII

2. Convert to X-units using the formula

$$\begin{aligned} X &= \mu + Z\sigma \\ &= 8 + (-.84) \cdot 5 = 3.8. \end{aligned}$$

So 20% of the values from a distribution with mean 8 and standard deviation 5 are less than 3.80.

Joint and Marginal Probability Distributions – I

Joint Probability Functions

- Suppose that X and Y are discrete random variables. The ***joint probability function*** is

$$P(x,y) = \Pr(X = x \cap Y = y),$$

which is simply used to express the probability that X takes the specific value x and simultaneously Y takes the value y, as a function of x and y. This should satisfy:

1. $0 \leq P(x,y) \leq 1$ for all x, y.
2. $\sum_x \sum_y P(x,y) = 1$, where the sum is over all values (x,y) that are assigned nonzero probabilities.

Joint and Marginal Probability Distributions – II

Joint Probability Functions

- For any random variables X and Y (discrete or continuous), the ***joint (bivariate) distribution function*** $F(x,y)$ is

$$F(x,y) = \Pr(X \leq x \cap Y \leq y).$$

This defines the probability that simultaneously X is less than x and Y is less than y.

Joint and Marginal Probability Distributions

Marginal Probability Functions

- Let X and Y be jointly discrete random variables with probability function $P(x,y)$. Then the ***marginal probability functions*** of X and Y, respectively, are given by

$$P_x(x) = \sum_{\text{all } y} P(x,y) \qquad P_y(y) = \sum_{\text{all } x} P(x,y)$$

- Let X and Y be jointly discrete random variables with probability function $P(x,y)$. The ***cumulative marginal probability functions***, denoted $F_x(x_0)$ and $G_y(y_0)$, show the probability that X is less than or equal to x_0 and that Y is less than or equal to y_0 respectively

$$\begin{aligned} F_x(x_0) &= \Pr(X \leq x_0) = \sum_{x \leq x_0} P_x(x), \\ G_y(y_0) &= \Pr(Y \leq y_0) = \sum_{y \leq y_0} P_y(y). \end{aligned}$$

Conditional Probability Distributions

- If X and Y are jointly discrete random variables with joint probability function $P(x, y)$ and marginal probability functions $P_x(x)$ and $P_y(y)$, respectively, then the conditional discrete probability function of Y given X is

$$P(y|x) = \Pr(Y = y|X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} = \frac{P(x, y)}{P_x(x)},$$

provided that $P_x(x) > 0$. Similarly,

$$P(x|y) = \frac{P(x, y)}{P_y(y)}, \text{ provided that } P_y(y) > 0$$

Statistical Independence

- Let X have distribution function $F_x(x)$, Y have distribution function $F_y(y)$, and X and Y have a joint distribution function $F(x, y)$. Then X and Y are said to be ***independent*** if and only if

$$F(x, y) = F_x(x) \cdot F_y(y),$$

for every pair of real numbers (x, y).

- Alternatively, the two random variables X and Y are independent if the conditional distribution of Y given X does not depend on X:

$$\Pr(Y = y|X = x) = \Pr(Y = y).$$

- We also define Y to be **mean independent** of X when the conditional mean of Y given X equals the unconditional mean of Y:

$$\mathrm{E}(Y = y|X = x) = \mathrm{E}(Y = y).$$

Conditional Moments

- If X and Y are any two discrete random variables, the ***conditional expectation*** of Y given that $X = x$, is defined to be

$$\mu_{Y|X} = \mathrm{E}(Y|X = x) = \sum_{\text{all } y} y \cdot P(y|x)$$

- If X and Y are any two discrete random variables, the ***conditional variance*** of Y given that $X = x$, is defined to be

$$\sigma^2_{Y|X} = \mathrm{E}[(Y - \mu_{Y|X})^2 | X = x] = \sum_{\text{all } y} (y - \mu_{Y|X})^2 \cdot P(y|x)$$

Joint and Marginal Distributions – I

Examples

- We are given the following data on the number of people attending AUEB this year.

	Subject of Study (Y)		
Sex (X)	***Economics*** (0)	***Finance*** (1)	***Systems*** (2)
Male (0)	40	10	30
Female (1)	30	20	70

1. What is the probability of selecting an individual that studies Finance?
2. What is the expected value of *Sex*?
3. What is the probability of choosing an individual that studies economics, given that it is a female?
4. Are *Sex* and *Subject* statistically independent?

Joint and Marginal Distributions – II

Examples

- **First step: Totals**

	Subject of Study (Y)			
Sex (X)	***Economics*** (0)	***Finance*** (1)	***Systems*** (2)	Total
Male (0)	40	10	30	**80**
Female (1)	30	20	70	**120**
Total	**70**	**30**	**100**	**200**

- **Second step: Probabilities**

	Subject of Study (Y)			
Sex (X)	***Economics*** (0)	***Finance*** (1)	***Systems*** (2)	Total
Male (0)	$40/200 = 0.20$	0.05	0.15	**0.40**
Female (1)	$30/200 = 0.15$	0.10	0.35	**0.60**
Total	$\mathbf{70/200 = 0.35}$	**0.15**	**0.50**	**1**

Joint and Marginal Distributions – III

Examples

- Answers:
  1. $\Pr(Y = 1) = 0.15$.
  2. $\mathrm{E}(X) = 0 \cdot 0.4 + 1 \cdot 0.6 = 0.6$
  3. $\Pr(Y = 0|X = 1) = 0.15/0.6 = 0.25$
  4. $\Pr(X = 0 \cap Y = 0) = 0.20 \neq \Pr(X = 0) \cdot \Pr(Y = 0) = 0.4 \cdot 0.35 = 0.14$. So *Sex* and *Subject* are not statistically independent.

  - The conditional mean of Y given $X = 0$ is

$$\mathrm{E}(Y|X = 0)$$
$$= \Pr(Y = 0|X = 0) \cdot 0 + \Pr(Y = 1|X = 0) \cdot 1 + \Pr(Y = 2|X = 0) \cdot 2$$
$$= \frac{0.20}{0.4} \cdot 0 + \frac{0.05}{0.4} \cdot 1 + \frac{0.15}{0.4} \cdot 2 = 0.875$$

Joint and Marginal Distributions – IV

Examples

- The conditional mean of Y given $X = 1$ is

$$\mathrm{E}(Y|X=1)$$
$$= \Pr(Y=0|X=1)\cdot 0 + \Pr(Y=1|X=1)\cdot 1 + \Pr(Y=2|X=1)\cdot 2$$
$$= \frac{0.15}{0.6}\cdot 0 + \frac{0.10}{0.6}\cdot 1 + \frac{0.35}{0.6}\cdot 2 = 0.80$$

Covariance, Correlation and Independence – I

Definition (Covariance)

If X and Y are random variables with means μ_x and μ_y, respectively, the ***covariance*** of X and Y is

$$\sigma_{XY} \equiv \operatorname{Cov}(X, Y) = \mathrm{E}\big[(X - \mu_x)(Y - \mu_y)\big].$$

- This can be found as

$$\operatorname{Cov}(X, Y) = \sum_{\text{all } x} \sum_{\text{all } y} (x - \mu_x)(y - \mu_y) \cdot P(x, y),$$

and an equivalent expression is

$$\operatorname{Cov}(X, Y) = \mathrm{E}[XY] - \mu_x \mu_y = \sum_{\text{all } x} \sum_{\text{all } y} xy \cdot P(x, y) - \mu_x \mu_y.$$

Covariance, Correlation and Independence – II

- The *covariance* measures the strength of the linear relationship between two variables.
- If two random variables are statistically independent, the covariance between them is 0. The converse is **not** necessarily true.

Covariance, Correlation and Independence – III

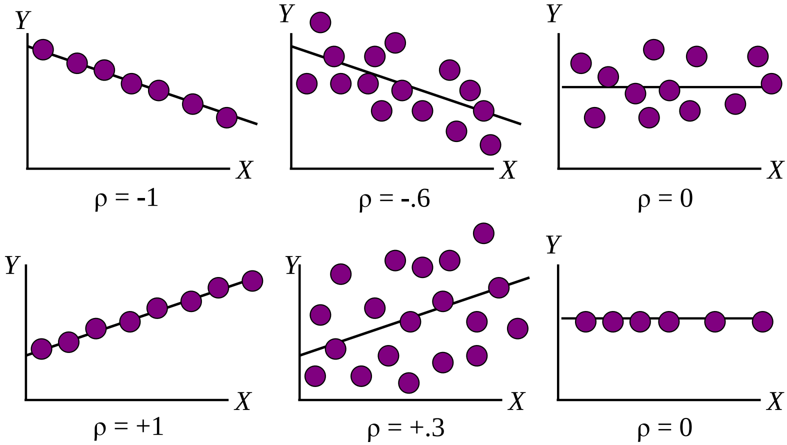
Definition (Correlation)

The correlation between X and Y is

$$\rho \equiv \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$$

- $\rho = 0 \Rightarrow$ no linear relationship between X and Y.
- $\rho > 0 \Rightarrow$ positive linear relationship between X and Y.
  - when X is high (low) then Y is likely to be high (low)
  - $\rho = +1 \Rightarrow$ perfect positive linear dependency
- $\rho < 0 \Rightarrow$ negative linear relationship between X and Y.
  - when X is high (low) then Y is likely to be low (high)
  - $\rho = -1 \Rightarrow$ perfect negative linear dependency

Covariance, Correlation and Independence – IV



Moments of Linear Combinations – I

- Let X and Y be two random variables with means μ_X and μ_Y, and variances σ_X^2 and σ_Y^2 and covariance $\mathrm{Cov}(X, Y)$. Take a linear combination of X and Y :

$$W = aX + bY.$$

Then,

$$\mathrm{E}(W) = \mathrm{E}(aX + bY) = a\mu_X + b\mu_Y, \text{ and}$$

$$\mathrm{Var}(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X, Y),$$

or using the correlation

$$\mathrm{Var}(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Corr}(X, Y)\sigma_X\sigma_Y$$

Moments of Linear Combinations – II

Example

If $a = 1$ and $b = -1$, $W = X - Y$ and

$$\begin{aligned} \mathrm{E}(W) &= \mathrm{E}(X - Y) = \mu_X - \mu_Y \\ \mathrm{Var}(W) &= \sigma_X^2 + \sigma_Y^2 - 2\mathrm{Cov}(X, Y) \\ &= \sigma_X^2 + \sigma_Y^2 - 2\mathrm{Corr}(X, Y)\sigma_X\sigma_Y \end{aligned}$$

Moments of Linear Combinations

Example 1: Normally Distributed Random Variables

- Two tasks must be performed by the same worker.
  - X = minutes to complete task 1; $\mu_X = 20, \sigma_X = 5$;
  - Y = minutes to complete task 2; $\mu_Y = 30, \sigma_Y = 8$;
  - X and Y are normally distributed and independent...

★ What is the mean and standard deviation of the time to complete both tasks?

- $W = X + Y$ (total time to complete both tasks). So

$$\begin{aligned} \mathrm{E}(W) &= \mu_X + \mu_Y = 20 + 30 = 50 \\ \mathrm{Var}(W) &= \sigma_X^2 + \sigma_Y^2 + \underbrace{2\mathrm{Cov}(X,Y)}_{=0,\text{ independence}} = 5^2 + 8^2 = 89 \\ \Rightarrow &\quad \sigma_W = \sqrt{89} \simeq 9.43 \end{aligned}$$

Linear Combinations Random Variables – I

Example 2: Portfolio Value

- The return per $\$1,000$ for two types of investments is given below

State of Economy		Investment Funds	
Prob	**Economic condition**	***Passive*** X	***Aggressive*** Y
0.2	Recession	$-\$25$	$-\$200$
0.5	Stable Economy	$+\$50$	$+\$60$
0.3	Growing Economy	$+\$100$	$+\$350$

- Suppose 40% of the portfolio (P) is in Investment X and 60% is in Investment Y. Calculate the portfolio return and risk.
  - Mean return for each fund investment

$$\begin{aligned} \mathrm{E}(X) &= \mu_X = (-25)(.2) + (50)(.5) + (100)(.3) = 50 \\ \mathrm{E}(Y) &= \mu_Y = (-200)(.2) + (60)(.5) + (350)(.3) = 95 \end{aligned}$$

Linear Combinations Random Variables – II

Example 2: Portfolio Value

- Standard deviations for each fund investment

$$
\begin{aligned}
\sigma_X &= \sqrt{(-25-50)^2(.2)+(50-50)^2(.5)+(100-50)^2(.3)} \\
&= 43.30 \\
\sigma_Y &= \sqrt{(-200-95)^2(.2)+(60-95)^2(.5)+(350-95)^2(.3)} \\
&= 193.71
\end{aligned}
$$

- The covariance between the two fund investments is

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= (-25-50)(-200-95)(.2) \\
&\quad +(50-50)(60-95)(.5) \\
&\quad +(100-50)(350-95)(.3) \\
&= 8250
\end{aligned}
$$

Linear Combinations Random Variables – III

Example 2: Portfolio Value

- So

$$\begin{aligned} \mathrm{E}(P) &= 0.4(50) + 0.6(95) = 77 \\ \sigma_P &= \sqrt{(.4)^2(43.30)^2 + (.6)^2(193.71)^2 + 2(.4)(.6)8250} \\ &= 133.04 \end{aligned}$$

The t-Distribution – I

- Let two independent random variables $Z \sim N(0,1)$ and $Y \sim \chi^2(n)$.[1] If Z and Y are independent, then

$$W = \frac{Z}{\sqrt{Y/n}} \sim t(n)$$

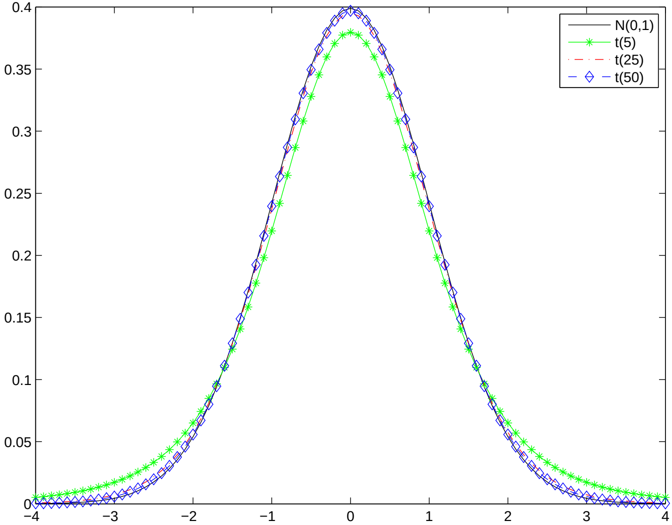
  - The PDF of t has only one parameter, n, is always positive and symmetric around zero.
  - Moreover it holds that

$$\mathrm{E}(W) = 0 \text{ for } n > 1; \quad \mathrm{Var}(W) = \frac{n}{n-2} \text{ for } n > 2$$

and for n large enough: $W \underset{n\to\infty}{\sim} N(0,1)$

[1] Let $Z_1, Z_2, \ldots, Z_n$ be independent r.v.s and $Z_i \sim N(0,1)$. Then $\Upsilon = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$.

The t-Distribution – II



Annex: Normal Approximation of Binomial – I

- Recall the binomial distribution, where we have n *independent trials* and the probability of success on any given trial = p.
- Let X be a binomial random variable ($X_i = 1$ if the ith trial is "success"):

$$\begin{aligned} \mathrm{E}(X) &= \mu = np \\ \mathrm{Var}(X) &= \sigma^2 = np(1-p) \end{aligned}$$

  - The shape of the binomial distribution is approximately normal if n is large

Annex: Normal Approximation of Binomial – II

- The normal is a good approximation to the binomial when $np(1-p) > 5$ (check that $np > 5$ and $n(1-p) > 5$ to be on the safe side). That is

$$Z = \frac{X - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}} = \frac{X - np}{\sqrt{np(1-p)}}.$$

- For instance, let X be the number of successes from n independent trials, each with probability of success p. Then

$$\Pr(a < X < b) = \Pr\left(\frac{a - np}{\sqrt{np(1-p)}} < Z < \frac{b - np}{\sqrt{np(1-p)}}\right)$$

Annex: Normal Approximation of Binomial – III

- **Example**: 40% of all voters support ballot proposition A. What is the probability that between 76 and 80 voters indicate support in a sample of n = 200?

$$\begin{aligned} \mathrm{E}(X) &= \mu = np = 200(0.40) = 80 \\ \mathrm{Var}(X) &= np(1-p) = 200(0.40)(1-0.40) = 48 \end{aligned}$$

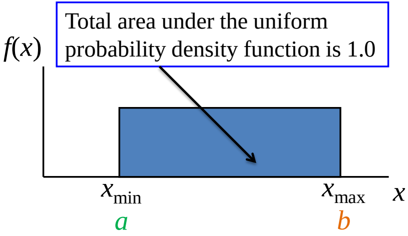
So

$$\begin{aligned} \Pr(76 < X < 80) &= \Pr\left(\frac{76-80}{\sqrt{48}} < Z < \frac{80-80}{\sqrt{48}}\right) \\ &= \Pr(-0.58 < Z < 0) \\ &= \Phi(0) - \Phi(-0.58) \\ &= 0.500 - 0.2810 = 0.219 \end{aligned}$$

Annex: Uniform Distribution – I

- The *uniform distribution* is a probability distribution that has *equal probabilities* for all possible outcomes of the random variable (where $x_{\min} = a$ and $x_{\max} = b$)

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \; ; \; F(x) \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & x \geq b \end{cases}$$



Annex: Uniform Distribution – II

- Moments uniform distribution

$$\mu = \frac{a+b}{2}; \qquad \sigma^2 = \frac{(b-a)^2}{12}$$

- **Example**: Uniform probability distribution over the range $2 \leq x \leq 6$. Then

$$f(x) = \frac{1}{6-2} = 0.25 \text{ for } 2 \leq x \leq 6$$

and

$$\begin{aligned} \mathrm{E}(X) &= \mu = \frac{a+b}{2} = \frac{2+6}{2} = 4 \\ \mathrm{Var}(X) &= \sigma^2 = \frac{(b-a)^2}{12} = \frac{(6-2)^2}{12} = 1.333 \end{aligned}$$

Annex: The χ^2 Distribution – I

- Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables and $Z_i \sim N(0,1)$. Then

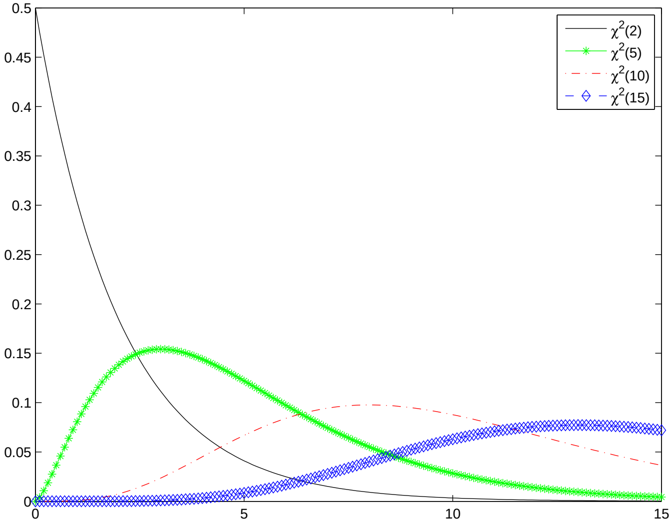
$$X = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$$

  - The PDF of χ^2 has only one parameter, n, is always positive and right asymmetric.
  - Moreover it holds that

$$\begin{aligned} \mathrm{E}(X) &= n; \text{ and} \\ \mathrm{Var}(X) &= 2n \end{aligned}$$

for $n \geq 2$.

Annex: The χ^2 Distribution – II



Annex: The F Distribution – I

- Let X and Y be two independent random variables, that are distributed as χ^2 : $X \sim \chi^2(n)$ and $Y \sim \chi^2(m)$. Then

$$W = \frac{X/n}{Y/m} \sim F(n, m)$$

  - The PDF of F has two parameters, n and m (the degrees of freedom of the numerator and the denominator); it is positive and right asymmetric.
  - Moreover it holds that if $W \sim F(n, m)$

$$\mathrm{E}(W) = \frac{m}{1-m}; \text{ for } m > 2.$$

Annex: The *F* Distribution – II