

Quantitative Methods Induction Course

Lecture 2: Differential & Integral Calculus with Optimization

Department of Accounting and Finance

MSc in Financial Management
Athens University of Economics and Business

Induction Program

Marginal Analysis, Dynamic Growth, Unconstrained & Constrained Optimization, Continuous Cash Flows

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Why Calculus in Financial Management?

- Financial decisions are fundamentally about **marginal trade-offs** and **optimal allocations**:
 - **Marginal Revenue & Cost**: What is the profit impact of selling one additional unit?
 - **Bond Sensitivity**: How does a 1 basis point shift in yield impact fixed income portfolio value? (**Duration & Convexity**).
 - **Continuous Compounding**: Modeling cash flows, stock prices, and option dynamics in continuous time (e^{rt}).
 - **Constrained Resource Allocation**: Maximizing investor utility or corporate enterprise value subject to budget and risk limits (**Lagrange Multipliers**).
 - **Asset Valuation**: Integrating continuous cash flow streams over project lifetimes.

Core Philosophy

Calculus transforms static financial accounting into dynamic decision models by quantifying *instantaneous rates of change*.

The Concept of a Limit

Definition of a Limit

Let $f(x)$ be defined on an open interval around x_0 . We say:

$$\lim_{x \rightarrow x_0} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to x_0 (from both left and right).

Key Limit Properties:

- $\lim[f(x) \pm g(x)] = \lim f(x) \pm \lim g(x)$
- $\lim[f(x) \cdot g(x)] = \lim f(x) \cdot \lim g(x)$
- $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$ (if $\lim g(x) \neq 0$)
- $\lim c \cdot f(x) = c \lim f(x)$

Common Exam Limit Trap

Consider $f(x) = \frac{1}{x^2}$ as $x \rightarrow 0$:

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

Because $x^2 > 0$ for all $x \neq 0$, both left and right limits shoot to $+\infty$. (Does not converge to a finite real number!)

Financial Application: Continuous Compounding

Compounding at Increasing Frequency

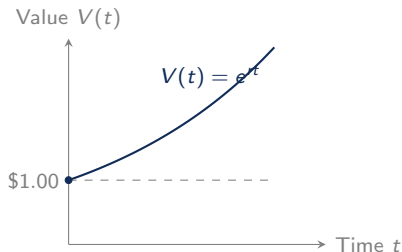
Suppose \$1 is invested at annual nominal interest rate r , compounded m times per year for t years:

$$V(m) = \left(1 + \frac{r}{m}\right)^{mt}$$

- Annual ($m = 1$): $(1 + r)^t$
- Semi-annual ($m = 2$): $(1 + r/2)^{2t}$
- Monthly ($m = 12$): $(1 + r/12)^{12t}$
- Daily ($m = 365$): $(1 + r/365)^{365t}$

The Limit as $m \rightarrow \infty$:

$$\lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^{mt} = \lim_{m \rightarrow \infty} \left[\left(1 + \frac{1}{m/r}\right)^{m/r}\right]^{rt} = e^{rt}$$



In continuous time, discounting is simply multiplication by e^{-rt} !

Continuity: Definition & Failure Modes

Definition of Continuity

A function $f(x)$ is continuous at a point x_0 if and only if three conditions are satisfied:

- 1 $f(x_0)$ is defined (point exists in the domain).
- 2 $\lim_{x \rightarrow x_0} f(x)$ exists (left limit equals right limit).
- 3 $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Standard Continuous Functions:

- Polynomials: $f(x) = ax^2 + bx + c$
- Exponentials: $f(x) = e^x, e^{kx}$
- Trigonometric: $\sin(x), \cos(x)$

Classic Discontinuous Function (Exam Favorite!)

$$f(x) = \frac{1}{x} \quad \text{at } x = 0$$

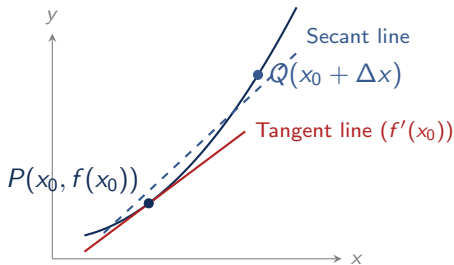
- $f(0)$ is undefined (division by zero).
- $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ and $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$.
- **Conclusion:** $f(x) = \frac{1}{x}$ is NOT continuous at

The Derivative: Geometric Definition

Definition as a Limit

The first derivative $f'(x)$ (or $\frac{df}{dx}$) represents the instantaneous rate of change:

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$



Geometric Takeaway:

- The secant slope $\frac{\Delta y}{\Delta x}$ connects two distinct points P and Q .
- As $\Delta x \rightarrow 0$, $Q \rightarrow P$, and the secant line rotates into the **tangent line**.
- The derivative $f'(x_0)$ is the **slope of the tangent line** at x_0 .

Differentiability vs. Continuity

Theorem (Differentiability Implies Continuity)

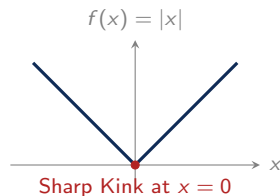
If a function $f(x)$ is differentiable at x_0 , then it is **necessarily continuous** at x_0 .

The Converse is FALSE!

Continuity does **not** guarantee differentiability. A function can be continuous but possess sharp "kinks" or vertical tangents.

The Classic Counterexample (Past Exam!):

$$f(x) = |x| \quad \text{at } x = 0$$



$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = +1, \quad \lim_{h \rightarrow 0^-} \frac{|0 + h| - 0}{h} = -1$$

Left and right derivatives disagree $\implies$ **Not differentiable** at $x = 0$.

Master Table of Derivatives

Function $f(x)$	First Derivative $f'(x)$	Financial Context / Meaning
Constant: c	0	Fixed overhead cost / sunk cost
Linear: $ax + b$	a	Constant marginal revenue / cost
Power Rule: cx^n	cnx^{n-1}	Production functions ($Q = K^\alpha L^\beta$)
Exponential: e^x	e^x	Continuous growth benchmark
Exponential: e^{kx}	ke^{kx}	Discount factor $\frac{d}{dt}(e^{-rt}) = -re^{-rt}$
Natural Log: $\ln(x)$	$\frac{1}{x} \ (x > 0)$	Proportional return: $d \ln(S_t) = \frac{dS_t}{S_t}$
Square Root: $\sqrt{x} = x^{1/2}$	$\frac{1}{2\sqrt{x}}$	Risk aversion utility: $U(W) = \sqrt{W}$
Reciprocal: $\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2}$	P/E multiple or yield sensitivity

Higher-Order Derivatives

The second derivative is the derivative of the first derivative:

$$f(x) = \ln(x) \implies f'(x) = \frac{1}{x} = x^{-1} \implies f''(x) = -x^{-2} = -\frac{1}{x^2} \quad (\text{Past Exam Question!})$$

Differentiation Rules for Combined Functions

Algebra of Derivatives

Let $u(x)$ and $v(x)$ be differentiable functions:

1 Sum/Difference Rule:

$$(u \pm v)' = u' \pm v'$$

2 Product Rule:

$$(u \cdot v)' = u'v + uv'$$

3 Quotient Rule:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad (v(x) \neq 0)$$

4 Chain Rule (Function of a Function):

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Chain Rule Example

Let $y = (3x^2 + 5)^4$. Here $u = 3x^2 + 5$ and $y = u^4$:

Instantaneous Rate of Growth & Log-Differentiation

Proportional Rate of Growth

The instantaneous proportional rate of change of a time-dependent variable $Y(t)$ is:

$$\text{Rate of Growth} = \frac{1}{Y(t)} \frac{dY}{dt} = \frac{d}{dt} \ln[Y(t)]$$

Proof via Chain Rule:

$$\frac{d}{dt} \ln[Y(t)] = \frac{1}{Y(t)} \cdot Y'(t) = \frac{Y'(t)}{Y(t)}$$

Properties of Growth Rates:

- If $Z = X \cdot Y \implies g_Z = g_X + g_Y$
- If $Z = X/Y \implies g_Z = g_X - g_Y$
- If $Z = X^a \implies g_Z = a \cdot g_X$

Financial Application

Suppose portfolio assets under management (AUM) evolve as $A(t) = A_0 e^{gt}$.

$$\ln A(t) = \ln A_0 + gt$$

$$\frac{d}{dt} \ln A(t) = 0 + g = \mathbf{g}$$

The slope of the logarithm of stock prices or revenues directly equals the **compound growth**

Monotonicity & Curvature: The Geometric Tests

First Derivative: Monotonicity

- If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is **strictly increasing**.
- If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is **strictly decreasing**.
- If $f'(x^*) = 0$, x^* is a **stationary (critical) point**.

Second Derivative: Curvature

- If $f''(x) > 0$ for all x , $f(x)$ is **strictly convex** (bowl-shaped $\cup$, slope increases).
- If $f''(x) < 0$ for all x , $f(x)$ is **strictly concave** (dome-shaped $\cap$, slope decreases).
- If $f''(x^*) = 0$ with sign change, x^* is an **inflection point**.

Concave ($f'' < 0 \Rightarrow \text{Max}$)



Convex ($f'' > 0 \Rightarrow \text{Min}$)



Unconstrained Optimization: First & Second Order Conditions

Necessary & Sufficient Conditions for Local Extrema

To optimize a smooth function $f(x)$:

1 First Order Condition (FOC - Necessary):

$$f'(x^*) = 0 \implies \text{Solve for candidate stationary points } x^*$$

2 Second Order Condition (SOC - Classification):

- If $f''(x^*) < 0 \implies x^*$ is a strict local maximum.
- If $f''(x^*) > 0 \implies x^*$ is a strict local minimum.
- If $f''(x^*) = 0 \implies$ Inconclusive (test higher derivatives or sign change).

Worked Math Example

Find the local extrema of $f(x) = x^3 - 3x + 5$:

- FOC: $f'(x) = 3x^2 - 3 = 0 \implies x^2 = 1 \implies x = +1$ or $x = -1$.
- SOC: $f''(x) = 6x$.
 - At $x = +1$: $f''(1) = 6(1) = +6 > 0 \implies$ **Local Minimum** at $(1, 3)$.

Global Extrema on a Closed Interval (Past Exam Problem!)

Past Exam Question

Which is the location of the minimum of $f(x) = 5x - 4x^2$ in the closed interval $[-5, 10]$?

- A) $x = 5/8$ B) $x = -5/8$ C) $x = 10$ D) $x = -5$

Methodical Step-by-Step Solution:

1 Find Interior Critical Points (FOC):

$$f'(x) = 5 - 8x = 0 \implies x^* = \frac{5}{8} = 0.625 \in [-5, 10]$$

$f''(x) = -8 < 0 \implies x^* = 5/8$ is a **Local Maximum**, NOT a minimum! (Exam Trap A avoided!)

2 Check the Boundaries of the Closed Interval $[-5, 10]$: On a closed bounded interval $[a, b]$, global extrema must occur at interior critical points or boundaries!

- At $x = 5/8$: $f(5/8) = 5(5/8) - 4(25/64) = 25/8 - 25/16 = 25/16 = +1.5625$ (Global Max).
- At left boundary $x = -5$:

$$f(-5) = 5(-5) - 4(-5)^2 = -25 - 4(25) = -25 - 100 = -125$$

- At right boundary $x = 10$:

$$f(10) = 5(10) - 4(10)^2 = 50 - 4(100) = 50 - 400 = -350$$

3 Comparison: Since $-350 < -125$, the global minimum is -350 , which occurs at $x = 10$!

Correct Answer: C ($x = 10$)

Never evaluate only the FOC; on closed intervals, always check boundary points!

Financial Application 1: Firm Profit Maximization

Microeconomic Foundation

Let Q denote output quantity. Total Profit is:

$$\Pi(Q) = TR(Q) - TC(Q)$$

where $TR(Q) = P(Q) \cdot Q$ is Total Revenue and $TC(Q)$ is Total Cost.

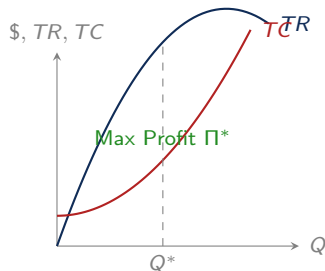
First Order Condition (FOC):

$$\frac{d\Pi}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ} = 0 \implies MR(Q^*) = MC(Q^*)$$

Second Order Condition (SOC):

$$\frac{d^2\Pi}{dQ^2} = \frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} < 0 \implies \frac{d(MC)}{dQ} > \frac{d(MR)}{dQ}$$

Economic Meaning: Marginal Cost must cut
Marginal Revenue from below!



Worked Profit Maximization Example

Market Parameters

A firm faces inverse demand $P(Q) = 250 - 20Q$ and cost function $TC(Q) = 30 + 10Q + 5Q^2$.

Step 1: Formulate Total Revenue & Marginal Revenue:

$$TR(Q) = P(Q) \cdot Q = (250 - 20Q)Q = 250Q - 20Q^2$$

$$MR(Q) = \frac{dTR}{dQ} = 250 - 40Q$$

Step 2: Formulate Marginal Cost:

$$MC(Q) = \frac{dTC}{dQ} = 10 + 10Q$$

Step 3: First Order Condition ($MR = MC$):

$$250 - 40Q = 10 + 10Q \implies 240 = 50Q \implies Q^* = 4.8 \text{ units}$$

Step 4: Second Order Condition:

$$\frac{d^2\Pi}{dQ^2} = MR'(Q) - MC'(Q) = -40 - 10 = -50 < 0 \implies \text{Strict Maximum!}$$

Financial Application 2: Bond Duration & Convexity

Bond Price as a Function of Yield y

$$P(y) = \sum_{t=1}^T \frac{C}{(1+y)^t} + \frac{M}{(1+y)^T}$$

1st Derivative $\Rightarrow$ Modified Duration (D^*):

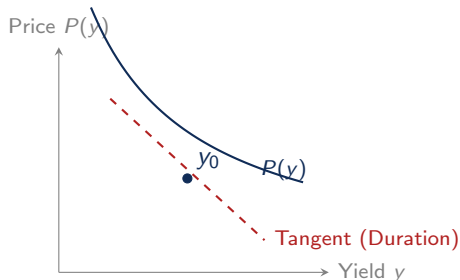
$$\frac{dP}{dy} = - \sum_{t=1}^T \frac{t \cdot CF_t}{(1+y)^{t+1}} \Rightarrow D^* = - \frac{1}{P} \frac{dP}{dy}$$

2nd Derivative $\Rightarrow$ Convexity (C):

$$\text{Conv} = \frac{1}{P} \frac{d^2P}{dy^2} > 0$$

Taylor Series Approximation:

$$\Delta P \approx -D^* \Delta y + \frac{1}{2} \text{Conv} (\Delta y)^2$$



Due to convexity, Duration alone underestimates price

Multivariable Calculus: Partial Derivatives

Definition of Partial Derivatives

For a function of multiple variables $z = f(x, y)$, the partial derivative with respect to x measures the rate of change keeping y fixed:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \quad \frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Theorem (Young's / Schwarz's Theorem)

If the second partial derivatives are continuous, cross-partials are **symmetric**:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad (f_{xy} = f_{yx})$$

The Hessian Matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

- Local Max: $\mathbf{H}$ is Negative Definite.
- Local Min: $\mathbf{H}$ is Positive Definite.

Constrained Optimization: The Lagrange Multiplier Method

The Economic Problem

Maximize (or minimize) objective function $f(x, y)$ subject to equality constraint $g(x, y) = c$.

Step 1: Construct the Lagrangian Function $\mathcal{L}$:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - c]$$

Step 2: First Order Conditions (FOC):

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \implies \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \implies \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(g(x, y) - c) = 0 \implies g(x, y) = c$$

Economic Intuition of Tangency

Dividing the first two equations: $\frac{f_x}{f_y} = \frac{g_x}{g_y} \implies$ Marginal Rate of Substitution (MRS) equals Price Ratio!

Economic Interpretation of the Lagrange Multiplier λ

The Shadow Price

The multiplier λ^* measures the **marginal increase in the optimal objective value** resulting from relaxing the constraint by one unit:

$$\lambda^* = \frac{df(x^*, y^*)}{dc}$$

In Corporate Finance:

- If $g(x, y) \leq B$ is a capital budget constraint, λ^* is the **marginal enterprise value** of \$1 additional capital.
- If $\lambda^* = 0$, the budget is not binding!

The Bordered Hessian Matrix $\bar{\mathbf{H}}$:

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

For a maximum with 2 variables and 1 constraint:
 $\det(\bar{\mathbf{H}}) > 0$.

Worked Constrained Optimization Example

Utility Maximization

Maximize $U(x, y) = x \cdot y$ subject to budget constraint $2x + 4y = 80$.

Step 1: Formulate Lagrangian:

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(2x + 4y - 80)$$

Step 2: First Order Conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda = 0 \implies y = 2\lambda$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 4\lambda = 0 \implies x = 4\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 80 - 2x - 4y = 0$$

Step 3: Solve for x^*, y^*, λ^* :

$$\frac{x}{y} = \frac{4\lambda}{2\lambda} = 2 \implies x = 2y$$

Substitute into budget: $2(2y) + 4y = 80 \implies 8y = 80 \implies y^* = 10, \quad x^* = 20.$

Indefinite Integrals: The Antiderivative

Definition of Indefinite Integral

The indefinite integral of $f(x)$ is the collection of all antiderivatives $F(x)$ such that $F'(x) = f(x)$:

$$\int f(x)dx = F(x) + C$$

where C is the arbitrary constant of integration.

Core Integration Rules:

- $\int kdx = kx + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int e^{kx} dx = \frac{1}{k}e^{kx} + C$

Recovering Cost Functions

Suppose marginal cost is $MC(Q) = 3Q^2 - 4Q + 5$ and fixed cost is $FC = 100$:

$$TC(Q) = \int (3Q^2 - 4Q + 5)dQ$$

$$TC(Q) = Q^3 - 2Q^2 + 5Q + C$$

Integration by Parts

Formula for Integration by Parts

Derived from the product rule of differentiation:

$$\int u \, dv = uv - \int v \, du \quad \text{or} \quad \int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Worked Example: Discounted Growing Cash Flow

Evaluate $\int xe^{-x}dx$:

- Choose $u = x \implies du = dx$.
- Choose $dv = e^{-x}dx \implies v = \int e^{-x}dx = -e^{-x}$.

Applying the formula:

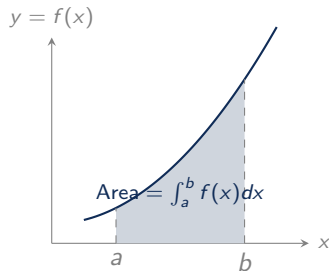
$$\int xe^{-x}dx = (x)(-e^{-x}) - \int (-e^{-x})dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C$$

Definite Integrals & Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$, then:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$



Past Exam Calculation:

$$\int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{2^3}{3} - 0 = \frac{8}{3}$$

$$\int_0^1 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_0^1 = \left(\frac{1}{3} + 1 \right) - 0 = \frac{4}{3}$$

Continuous Cash Flow Valuation

Continuous Present Value of Cash Flow Stream $C(t)$

If a financial asset generates a continuous stream of cash flows $C(t)$ from $t = 0$ to $t = T$, discounted at continuous rate r :

$$PV = \int_0^T C(t)e^{-rt} dt$$

Constant Perpetual Cash Flow ($C(t) = C$ and $T \rightarrow \infty$)

$$PV = \int_0^{\infty} Ce^{-rt} dt = C \left[\frac{e^{-rt}}{-r} \right]_0^{\infty} = C \left(0 - \frac{1}{-r} \right) = \frac{C}{r}$$

This rigorously derives the famous **Perpetuity Formula** using integral calculus!

Exam Practice 1: Derivatives & Continuity

Question 1 (Past Exam)

What is the derivative of $f(x) = x^2 + 3x + 2$?

- A) $2x + 3$ B) $x + 3$ C) $2x$ D) $x^2 + 3$

Exam Practice 1: Derivatives & Continuity

Question 1 (Past Exam)

What is the derivative of $f(x) = x^2 + 3x + 2$?

- A) $2x + 3$ B) $x + 3$ C) $2x$ D) $x^2 + 3$

Answer: A

Using power rule: $\frac{d}{dx}(x^2) = 2x$, $\frac{d}{dx}(3x) = 3$, $\frac{d}{dx}(2) = 0$. Sum = $2x + 3$.

Question 2 (Past Exam)

Which of the following functions is not continuous at $x = 0$?

- A) $f(x) = x^2$ B) $f(x) = \frac{1}{x}$ C) $f(x) = \sin(x)$ D) $f(x) = \cos(x)$

Exam Practice 1: Derivatives & Continuity

Question 1 (Past Exam)

What is the derivative of $f(x) = x^2 + 3x + 2$?

- A) $2x + 3$ B) $x + 3$ C) $2x$ D) $x^2 + 3$

Answer: A

Using power rule: $\frac{d}{dx}(x^2) = 2x$, $\frac{d}{dx}(3x) = 3$, $\frac{d}{dx}(2) = 0$. Sum = $2x + 3$.

Question 2 (Past Exam)

Which of the following functions is not continuous at $x = 0$?

- A) $f(x) = x^2$ B) $f(x) = \frac{1}{x}$ C) $f(x) = \sin(x)$ D) $f(x) = \cos(x)$

Answer: B

$f(0) = \frac{1}{0}$ is undefined and limits diverge to $\pm\infty$.

Exam Practice 2: Second Derivatives & Integrals

Question 3 (Past Exam)

If $f(x) = \ln(x)$, what is $f''(x)$?

- A) $\frac{1}{x^2}$ B) $-\frac{1}{x^2}$ C) $\frac{1}{x}$ D) $-\frac{1}{x}$

Exam Practice 2: Second Derivatives & Integrals

Question 3 (Past Exam)

If $f(x) = \ln(x)$, what is $f''(x)$?

- A) $\frac{1}{x^2}$ B) $-\frac{1}{x^2}$ C) $\frac{1}{x}$ D) $-\frac{1}{x}$

Answer: B

$$f'(x) = x^{-1} \implies f''(x) = (-1)x^{-2} = -\frac{1}{x^2}.$$

Question 4 (Past Exam)

What is the value of the definite integral $\int_0^1 (x^2 + 1)dx$?

- A) $\frac{2}{3}$ B) $\frac{4}{3}$ C) $\frac{1}{3}$ D) $\frac{5}{3}$

Exam Practice 2: Second Derivatives & Integrals

Question 3 (Past Exam)

If $f(x) = \ln(x)$, what is $f''(x)$?

- A) $\frac{1}{x^2}$ B) $-\frac{1}{x^2}$ C) $\frac{1}{x}$ D) $-\frac{1}{x}$

Answer: B

$$f'(x) = x^{-1} \implies f''(x) = (-1)x^{-2} = -\frac{1}{x^2}.$$

Question 4 (Past Exam)

What is the value of the definite integral $\int_0^1 (x^2 + 1)dx$?

- A) $\frac{2}{3}$ B) $\frac{4}{3}$ C) $\frac{1}{3}$ D) $\frac{5}{3}$

Answer: B

$$\left[\frac{x^3}{3} + x \right]_0^1 = \left(\frac{1}{3} + 1 \right) - 0 = \frac{4}{3}.$$

Exam Practice 3: Differentiability & Area

Question 5 (Past Exam)

If a function is differentiable at a point, it must also be:

- A) Continuous at that point B) Discontinuous at that point
- C) Have a maximum or minimum at that point D) None of the above

Exam Practice 3: Differentiability & Area

Question 5 (Past Exam)

If a function is differentiable at a point, it must also be:

- A) Continuous at that point B) Discontinuous at that point
- C) Have a maximum or minimum at that point D) None of the above

Answer: A

Differentiability is a strictly stronger condition that implies continuity.

Question 6 (Past Exam)

Which of the following functions is not differentiable at $x = 0$?

- A) $f(x) = x^2$ B) $f(x) = |x|$ C) $f(x) = \exp(x)$ D) $f(x) = \tan(x)$

Exam Practice 3: Differentiability & Area

Question 5 (Past Exam)

If a function is differentiable at a point, it must also be:

- A) Continuous at that point B) Discontinuous at that point
- C) Have a maximum or minimum at that point D) None of the above

Answer: A

Differentiability is a strictly stronger condition that implies continuity.

Question 6 (Past Exam)

Which of the following functions is not differentiable at $x = 0$?

- A) $f(x) = x^2$ B) $f(x) = |x|$ C) $f(x) = \exp(x)$ D) $f(x) = \tan(x)$

Answer: B

$f(x) = |x|$ has a sharp corner at $x = 0$ with left slope -1 and right slope $+1$.

Lecture 2 Summary & Key Formulas

Key Calculus Formulas:

- Continuous compounding:
 $\lim_{m \rightarrow \infty} (1 + r/m)^{mt} = e^{rt}$
- Growth rate: $g = \frac{d}{dt} \ln Y(t) = \frac{Y'(t)}{Y(t)}$
- Power rule: $\frac{d}{dx} x^n = nx^{n-1}$
- Log derivative: $\frac{d}{dx} \ln x = \frac{1}{x}$
- Product rule: $(uv)' = u'v + uv'$
- Quotient rule: $(u/v)' = \frac{u'v - uv'}{v^2}$
- Fundamental Thm: $\int_a^b f(x)dx = F(b) - F(a)$

Optimization & Financial Rules:

- Unconstrained extrema: $f'(x^*) = 0$;
 $f'' < 0 \implies \text{Max}$, $f'' > 0 \implies \text{Min}$.
- Profit Max: $MR(Q^*) = MC(Q^*)$.
- Bond Duration: $D^* = -\frac{1}{P} \frac{dP}{dy}$, Convexity > 0 .
- Constrained optimization: $\mathcal{L} = f - \lambda(g - c)$.
- Multiplier $\lambda = \frac{df^*}{dc}$ is the shadow price.
- Perpetuity PV: $\int_0^\infty Ce^{-rt} dt = \frac{C}{r}$.

Next Lecture: Probability Theory & Statistical Inference