

Quantitative Methods Induction Course

Lecture 1: Linear Algebra for Financial Management

Department of Accounting and Finance

MSc in Financial Management
Athens University of Economics and Business

Induction Program

Foundations for Asset Pricing, Portfolio Mathematics, Econometrics & Risk Analysis

Induction Roadmap: 3 Three-Hour Lectures

Lecture 1: Algebra

Focus: Structural Models

- Vectors & Matrix algebra
- Portfolio mathematics
- Determinants & Inverses
- Systems of equations
- Arbitrage & State prices
- Eigenvalues & Definiteness

Lecture 2: Calculus

Focus: Dynamic Optimization

- Limits & Compounding
- Marginal change & Rates
- Unconstrained extrema
- Multivariable calculus
- Constrained optimization
- Continuous cash flows

Lecture 3: Statistics

Focus: Risk & Uncertainty

- Probability & Bayes' rule
- Random variables & Fat tails
- Parametric distributions
- Central Limit Theorem
- Confidence intervals
- Hypothesis testing

Pedagogical Goal

Equip financial management students with the technical fluency and conceptual intuition required for corporate finance, asset pricing, derivatives, and financial econometrics.

Lecture 1 Table of Contents

- 1 Vectors, Matrices & Elementary Operations
- 2 Portfolio Mathematics in Matrix Notation
- 3 Determinants & Matrix Inversion
- 4 Systems of Linear Equations & State-Contingent Pricing
- 5 Quadratic Forms, Definiteness & Eigenvalues
- 6 Review & Exam Practice Problems

Why Linear Algebra in Financial Management?

- Modern finance rarely deals with a single asset or variable in isolation:
 - **Asset Portfolios:** Simultaneous management of hundreds of equities, bonds, and derivatives.
 - **Risk Management:** Modeling interdependencies via multi-asset variance-covariance matrices.
 - **Asset Pricing Models:** Factor models (CAPM, Fama-French) expressed as matrix systems.
 - **No-Arbitrage Pricing:** Determining state-contingent prices across economic scenarios.

Core Paradigm

Linear algebra provides a compact, powerful language to represent and solve multi-dimensional economic systems that would otherwise require hundreds of simultaneous scalar equations.

Vectors: Financial Representation

Mathematical Definition

An n -dimensional column vector $\mathbf{x} \in \mathbb{R}^n$ is an ordered collection of n real numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}' = [x_1 \quad x_2 \quad \dots \quad x_n]$$

$\mathbf{x}'$ (or $\mathbf{x}^T$) denotes the **transpose** (row vector).

Financial Translation

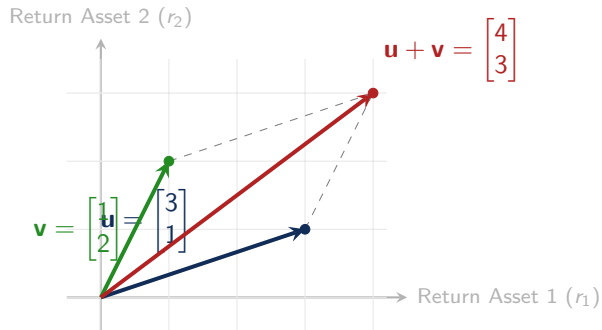
Portfolio Weights: Fraction of wealth invested in n securities:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad \text{with } \sum_{i=1}^n w_i = 1$$

Expected Returns:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} 8\% \\ 12\% \\ 5\% \end{bmatrix}$$

Geometric Intuition: Vector Space in $\mathbb{R}^2$



Key Vector Operations:

- 1 **Vector Addition:** Component-wise addition.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

- 2 **Scalar Multiplication:** Scaling risk/exposure.

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$

- 3 **Linear Combination:**

$$\mathbf{p} = w_1\mathbf{u} + w_2\mathbf{v}$$

Matrix Concept & Special Matrix Types

Matrix Definition

A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a rectangular array with m rows and n columns:

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Crucial Matrix Architectures in Finance:

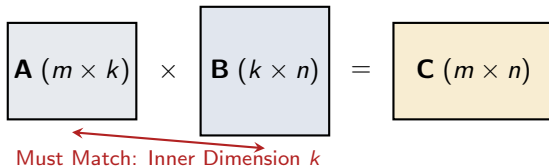
- **Square Matrix:** $m = n$ (e.g., asset correlation matrix).
- **Symmetric Matrix:** $\mathbf{A} = \mathbf{A}'$, so $a_{ij} = a_{ji}$ (e.g., covariance matrix $\mathbf{\Sigma}$).
- **Diagonal Matrix:** $a_{ij} = 0$ for all $i \neq j$ (uncorrelated idiosyncratic risks).
- **Identity Matrix ($\mathbf{I}_n$):** Diagonal elements are 1, off-diagonals are 0. Acts as the number 1 ($\mathbf{AI} = \mathbf{IA} = \mathbf{A}$).
- **Zero Matrix ($\mathbf{0}$):** All elements are 0.

Conformability for Matrix Multiplication

Conformability Condition

Two matrices **A** and **B** can be multiplied to form **AB** if and only if the number of columns in **A** equals the number of rows in **B**:

$$\begin{matrix} \mathbf{A} \\ (m \times k) \end{matrix} \times \begin{matrix} \mathbf{B} \\ (k \times n) \end{matrix} = \begin{matrix} \mathbf{C} \\ (m \times n) \end{matrix}$$



- Each element c_{ij} is the **inner product** (dot product) of row i of **A** and column j of **B**:

$$c_{ij} = \sum_{l=1}^k a_{il} b_{lj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj}$$

Fundamental Matrix Laws

■ Associative:

$$(AB)C = A(BC)$$

■ Distributive:

$$A(B + C) = AB + AC$$

■ Transpose of Product (Reversal Rule):

$$(AB)' = B'A'$$

■ Identity:

$$AI = IA = A$$

WARNING: Non-Commutative!

In scalar arithmetic: $a \cdot b = b \cdot a$.

In matrix algebra:

$$AB \neq BA \quad (\text{in general})$$

Numerical Counterexample: Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \neq BA = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$$

Order of operations is paramount in econometric derivations!

Portfolio Expected Return in Matrix Form

- Consider an investment portfolio across N risky assets.
- Let $\mathbf{w} = [w_1, w_2, \dots, w_N]'$ be the vector of asset weights with $\mathbf{w}'\mathbf{1} = \sum_{i=1}^N w_i = 1$.
- Let $\mathbf{r} = [r_1, r_2, \dots, r_N]'$ be the vector of expected returns: $\mathbb{E}[R_i] = r_i$.

Portfolio Expected Return Formulation

The expected portfolio return is the inner product of weights and returns:

$$\mathbb{E}[R_p] = \mathbf{w}'\mathbf{r} = \begin{bmatrix} w_1 & w_2 & \dots & w_N \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} = w_1 r_1 + w_2 r_2 + \dots + w_N r_N = \sum_{i=1}^N w_i r_i$$

Numerical Mini-Example:

$$\mathbf{w} = \begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 0.08 \\ 0.15 \end{bmatrix} \implies \mathbb{E}[R_p] = \begin{bmatrix} 0.40 & 0.60 \end{bmatrix} \begin{bmatrix} 0.08 \\ 0.15 \end{bmatrix} = (0.40)(0.08) + (0.60)(0.15) = 0.122 \quad (12.2\%)$$

The Variance-Covariance Matrix (Σ)

Structure of the Covariance Matrix

Let $\sigma_{ij} = \text{Cov}(R_i, R_j)$ and $\sigma_i^2 = \sigma_{ii} = \text{Var}(R_i)$. The $N \times N$ matrix is:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_N^2 \end{bmatrix}$$

Key Properties of Σ :

- 1 Symmetric:** $\sigma_{ij} = \sigma_{ji}$, therefore $\Sigma = \Sigma'$.
- 2 Main Diagonal:** Contains individual asset variances $\sigma_i^2 > 0$.
- 3 Off-Diagonal:** Contains pairwise covariances $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

Decomposition into Correlation

$$\Sigma = \mathbf{D}\mathbf{P}\mathbf{D}$$

where $\mathbf{D} = \text{diag}(\sigma_1, \dots, \sigma_N)$ is the volatility matrix, and $\mathbf{P}$ is the correlation matrix with 1s on diagonal.

Portfolio Variance: The Quadratic Form $\mathbf{w}'\Sigma\mathbf{w}$

Portfolio Variance Derivation

$$\sigma_p^2 = \text{Var}(\mathbf{w}'\mathbf{R}) = \mathbf{w}'\Sigma\mathbf{w} = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$$

Step-by-Step Multiplication for $N = 2$:

$$\mathbf{w}'\Sigma = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} w_1\sigma_1^2 + w_2\sigma_{12} & w_1\sigma_{12} + w_2\sigma_2^2 \end{bmatrix}$$

Multiplying this 1×2 row by the 2×1 column vector $\mathbf{w}$:

$$\sigma_p^2 = (\mathbf{w}'\Sigma)\mathbf{w} = \begin{bmatrix} w_1\sigma_1^2 + w_2\sigma_{12} & w_1\sigma_{12} + w_2\sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2$$

Insight

Matrix notation condenses N^2 terms into three symbols: $\mathbf{w}'\Sigma\mathbf{w}$.

Comprehensive Worked Example: 2-Asset Portfolio

Market Parameters

Stock A (Equities) and Stock B (Bonds):

$$\mathbf{w} = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 0.0400 & 0.0060 \\ 0.0060 & 0.0100 \end{bmatrix}$$

Note: $\sigma_A = \sqrt{0.04} = 20\%$, $\sigma_B = \sqrt{0.01} = 10\%$, $\rho_{AB} = \frac{0.0060}{(0.20)(0.10)} = 0.30$.

Step 1: Expected Return:

$$\mathbb{E}[R_p] = \mathbf{w}'\mathbf{r} = 0.60(0.10) + 0.40(0.05) = 0.060 + 0.020 = \mathbf{0.080} \text{ (8.0\%)}$$

Step 2: Intermediate Vector $\mathbf{w}'\mathbf{\Sigma}$:

$$\mathbf{w}'\mathbf{\Sigma} = \begin{bmatrix} 0.60 & 0.40 \end{bmatrix} \begin{bmatrix} 0.0400 & 0.0060 \\ 0.0060 & 0.0100 \end{bmatrix} = \begin{bmatrix} 0.0240 + 0.0024 & 0.0036 + 0.0040 \end{bmatrix} = \begin{bmatrix} 0.0264 & 0.0076 \end{bmatrix}$$

Step 3: Portfolio Variance & Volatility:

$$\sigma_p^2 = \begin{bmatrix} 0.0264 & 0.0076 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix} = (0.0264)(0.60) + (0.0076)(0.40) = \mathbf{0.01888}$$

$$\sigma_p = \sqrt{0.01888} \approx \mathbf{13.74\%} < \text{Weighted avg } (0.6 \times 20\% + 0.4 \times 10\% = 16.0\%) \implies \mathbf{Diversification!}$$

The Determinant: Definition & Geometric Meaning

Definition

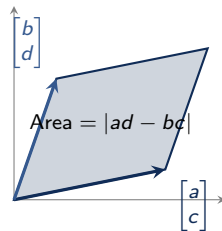
The determinant $\det(\mathbf{A})$ or $|\mathbf{A}|$ is a scalar value computed from a square matrix that encodes its scaling and invertibility properties.

Case 2×2 Matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \det(\mathbf{A}) = ad - bc$$

Geometric Meaning:

- $|\det(\mathbf{A})|$ represents the area of the parallelogram formed by the column vectors.
- If $\det(\mathbf{A}) = 0$, the parallelogram collapses into a single line $\implies$ vectors are collinear!



Determinant of a 3×3 Matrix: Laplace Expansion

Laplace Minor & Cofactor Expansion

For any $n \times n$ matrix $\mathbf{A}$, the cofactor of element a_{ij} is:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the $(n-1) \times (n-1)$ determinant formed by deleting row i and column j .

Expanding along the first row:

$$\det(\mathbf{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Explicit Formulation for 3×3 :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Numerical Calculation of a 3×3 Determinant

Worked Example

Compute the determinant of the factor loading matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

Solution via First Row Expansion:

$$\det(\mathbf{A}) = 2 \cdot C_{11} + 1 \cdot C_{12} + 0 \cdot C_{13}$$

- $M_{11} = \begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} = (3)(4) - (-1)(2) = 12 + 2 = 14 \implies C_{11} = (+1)(14) = 14$
- $M_{12} = \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} = (1)(4) - (-1)(0) = 4 - 0 = 4 \implies C_{12} = (-1)(4) = -4$
- C_{13} is multiplied by 0, so no need to evaluate.

$$\det(\mathbf{A}) = 2(14) + 1(-4) + 0 = 28 - 4 = \mathbf{24}$$

Properties of Determinants (Exam Essentials)

Property	Mathematical Rule	Financial Consequence
Transpose	$\det(\mathbf{A}') = \det(\mathbf{A})$	Variance-covariance symmetry
Product Rule	$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$	Multi-stage transformations
Scalar Multiple	$\det(c\mathbf{A}) = c^n \det(\mathbf{A})$ for $n \times n$	Exponent n is crucial!
Inverse	$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$	Invertible if and only if $\det(\mathbf{A}) \neq 0$
Row of Zeros	$\det(\mathbf{A}) = 0$	Redundant asset / arbitrage
Identical Rows	$\det(\mathbf{A}) = 0$	Collinear factors / multicollinearity
Row Operation	$R_j \leftarrow R_j + cR_i$ leaves det unchanged	Gaussian elimination preserves $ \mathbf{A} $
Row Swap	Swapping two rows multiplies det by -1	Parity alternation

Singularity Rule

$\det(\mathbf{A}) = 0 \iff$ Matrix is singular $\iff$ Columns are linearly dependent $\iff$ No unique inverse exists.

The Inverse Matrix: Concept & Conditions

Definition of Inverse

For a square $n \times n$ matrix $\mathbf{A}$, its inverse $\mathbf{A}^{-1}$ satisfies:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

Conditions for Invertibility:

- 1 $\mathbf{A}$ must be **square** ($n \times n$). Non-square matrices have no standard inverse!
- 2 $\mathbf{A}$ must be **non-singular**: $\det(\mathbf{A}) \neq 0$.
- 3 $\mathbf{A}$ must have **full rank**: $\text{rank}(\mathbf{A}) = n$ (all rows/columns linearly independent).

Properties of Inverses

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (Reversal rule!)
- $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$
- $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$ for $c \neq 0$

Computing Inverses: The Adjugate Formula

General Inversion Formula

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}) = \frac{1}{\det(\mathbf{A})} \mathbf{C}'$$

where $\mathbf{C} = [C_{ij}]$ is the cofactor matrix and $\text{adj}(\mathbf{A}) = \mathbf{C}'$ is the transpose of the cofactor matrix.

Explicit 2×2 Formula (Master this!):

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Rule: Swap main diagonal, negate off-diagonals, divide by determinant.

Worked 2×2 Example

$$\text{Let } \mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}.$$

$$\det(\mathbf{A}) = (4)(2) - (2)(3) = 8 - 6 = 2$$

$$\mathbf{A}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1.5 & 2 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 4-3 & -4+4 \\ 3-3 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

General Linear System (m equations in n unknowns)

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \iff \mathbf{Ax} = \mathbf{b}$$

- $\mathbf{A} \in \mathbb{R}^{m \times n}$: Coefficient matrix.
- $\mathbf{x} \in \mathbb{R}^{n \times 1}$: Vector of unknown variables (e.g., portfolio weights, prices).
- $\mathbf{b} \in \mathbb{R}^{m \times 1}$: Vector of constants (e.g., target returns, observed asset prices).
- **Augmented Matrix:** $[\mathbf{A} \mid \mathbf{b}] \in \mathbb{R}^{m \times (n+1)}$.

Existence and Uniqueness: Rouché-Capelli Theorem

Theorem (**Classification of Solutions for $Ax = b$**)

Let $\text{rank}(A)$ be the number of linearly independent rows/columns:

1 Unique Solution:

$$\text{rank}(A) = \text{rank}([A \mid b]) = n \quad (\text{if } m = n, \iff \det(A) \neq 0)$$

The solution is directly given by $x = A^{-1}b$.

2 Infinitely Many Solutions:

$$\text{rank}(A) = \text{rank}([A \mid b]) = k < n$$

There are $n - k$ free parameters (under-determined system / incomplete market).

3 No Solution (Inconsistent):

$$\text{rank}(A) < \text{rank}([A \mid b])$$

Equations contradict each other $\implies$ Arbitrage opportunity in financial markets!

Gaussian Elimination & Elementary Row Operations

Three Elementary Row Operations (EROs)

These operations alter the system's appearance but **preserve its solution set**:

- 1 Multiply a row by a non-zero scalar: $R_i \leftarrow cR_i$ ($c \neq 0$).
- 2 Add a multiple of one row to another row: $R_j \leftarrow R_j + cR_i$.
- 3 Swap two rows: $R_i \leftrightarrow R_j$.

Original Augmented Matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$$

$\xrightarrow{\text{EROs}}$

Reduced Row Echelon Form (RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1^* \\ 0 & 1 & 0 & x_2^* \\ 0 & 0 & 1 & x_3^* \end{array} \right]$$

Financial Application: Arrow-Debreu State Pricing

Economic Setting

An economy has two future states of the world at $t = 1$: State 1 (Boom) and State 2 (Recession). Two securities trade at $t = 0$:

- **Stock (j)**: Current price $p_j = \$8$. Payoff: \$10 in Boom, \$4 in Recession.
- **Bond (k)**: Current price $p_k = \$9$. Payoff: \$10 in Boom, \$10 in Recession.

Goal: Find the Arrow-Debreu state prices q_1, q_2 (price today of \$1 received in state s).

$$10q_1 + 4q_2 = 8 \quad (\text{Stock Pricing Equation})$$

$$10q_1 + 10q_2 = 9 \quad (\text{Bond Pricing Equation})$$

In matrix form $\mathbf{Dq} = \mathbf{p}$:

$$\begin{bmatrix} 10 & 4 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

Solving for State Prices & No-Arbitrage

$$\mathbf{D} = \begin{bmatrix} 10 & 4 \\ 10 & 10 \end{bmatrix} \implies \det(\mathbf{D}) = (10)(10) - (4)(10) = 100 - 40 = 60 \neq 0$$

$$\mathbf{D}^{-1} = \frac{1}{60} \begin{bmatrix} 10 & -4 \\ -10 & 10 \end{bmatrix}$$

Multiplying by the asset price vector $\mathbf{p} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 10 & -4 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 80 - 36 \\ -80 + 90 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 44 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{11}{15} \\ \frac{1}{6} \end{bmatrix} \approx \begin{bmatrix} \$0.733 \\ \$0.167 \end{bmatrix}$$

Financial Insights & Arbitrage-Free Checks

- 1 **No-Arbitrage Condition:** $q_1 > 0$ and $q_2 > 0$ (all state prices strictly positive!).
- 2 **Risk-Free Rate:** A riskless \$1 payoff costs $q_1 + q_2 = \frac{44}{60} + \frac{10}{60} = \frac{54}{60} = \0.90 .

$$1 + r_f = \frac{1}{q_1 + q_2} = \frac{1}{0.90} \approx 1.111 \implies r_f = 11.11\%$$

- 3 **Pricing Any Derivative:** A call option paying \$5 in Boom and \$0 in Recession must cost:

$$P_{\text{call}} = 5q_1 + 0q_2 = 5(0.733) = \$3.67$$

Quadratic Forms & Matrix Definiteness

Definition of Quadratic Form

For a symmetric $n \times n$ matrix $\mathbf{A}$ and non-zero vector $\mathbf{x} \in \mathbb{R}^n$, the quadratic form is:

$$Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j$$

Classification	Condition for all $\mathbf{x} \neq 0$	Financial Meaning
Positive Definite (PD)	$\mathbf{x}'\mathbf{A}\mathbf{x} > 0$	Non-degenerate covariance ($\sigma_p^2 > 0$)
Positive Semi-Definite (PSD)	$\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$	Arbitrage-free covariance ($\sigma_p^2 \geq 0$)
Negative Definite (ND)	$\mathbf{x}'\mathbf{A}\mathbf{x} < 0$	Strict profit maximum SOC
Indefinite	Changes sign depending on $\mathbf{x}$	Saddle point in optimization

Key Theorem for Finance

Every valid variance-covariance matrix $\mathbf{\Sigma}$ **must be positive semi-definite** (since variance cannot be negative: $\sigma_p^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \geq 0$). If no asset is redundant, $\mathbf{\Sigma}$ is strictly **positive definite** and therefore **invertible**!

Eigenvalues & Eigenvectors: Concept

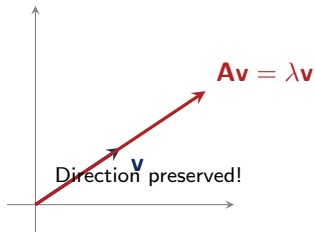
The Characteristic Equation

Let $\mathbf{A}$ be an $n \times n$ matrix. A non-zero vector $\mathbf{v}$ is an **eigenvector** and scalar λ is its corresponding **eigenvalue** if:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \iff (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

For a non-trivial solution ($\mathbf{v} \neq \mathbf{0}$), the matrix $(\mathbf{A} - \lambda\mathbf{I})$ must be singular:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (\text{Characteristic Equation})$$



Financial Application: PCA

In financial asset pricing, Principal Component Analysis (PCA) decomposes Σ :

- $\mathbf{v}_1$: Market factor (Level)
- $\mathbf{v}_2$: Term structure (Slope)
- $\mathbf{v}_3$: Curvature factor

Step-by-Step Eigenvalue Calculation

Worked Example

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Step 1: Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} = 0$$

$$(-5 - \lambda)(-2 - \lambda) - (2)(2) = (\lambda + 5)(\lambda + 2) - 4 = \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = 0$$

Factoring the quadratic equation:

$$(\lambda + 6)(\lambda + 1) = 0 \implies \lambda_1 = -6, \quad \lambda_2 = -1$$

Definiteness Check

Both eigenvalues are strictly negative ($\lambda_1 < 0, \lambda_2 < 0$) $\implies \mathbf{A}$ is **Negative Definite**!

Finding the Eigenvectors

For $\lambda_1 = -6$:

$$(\mathbf{A} - (-6)\mathbf{I})\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + 2v_2 = 0$$

$$v_1 = -2v_2 \implies \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$:

$$(\mathbf{A} - (-1)\mathbf{I})\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -4v_1 + 2v_2 = 0$$

$$v_2 = 2v_1 \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Orthogonality Check

Notice $\mathbf{v}_1 \cdot \mathbf{v}_2 = (-2)(1) + (1)(2) = 0$. Eigenvectors of symmetric matrices are always **orthogonal**!

Exam Practice 1: Matrix Transpose & Operations

Question 1 (Past Exam)

If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, what is $\mathbf{A}'$ (the transpose of $\mathbf{A}$)?

- A) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ C) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ D) $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Exam Practice 1: Matrix Transpose & Operations

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Answer: B

The transpose exchanges rows and columns: Row 1 $[1, 2]$ becomes Column 1, and Row 2 $[3, 4]$ becomes Column 2.

Exam Practice 2: Conformability for Multiplication

Question 2 (Past Exam)

If two matrices **A** and **B** are conformable for multiplication **AB**, then:

- A) The number of columns in **A** is equal to the number of rows in **B**.
- B) The number of rows in **A** is equal to the number of columns in **B**.
- C) The number of rows in **A** is equal to the number of rows in **B**.
- D) The number of columns in **A** is equal to the number of columns in **B**.

Exam Practice 2: Conformability for Multiplication

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- D) The number of columns in **A** is equal to the number of columns in **B**.

Answer: A

Multiplication requires inner dimensions to match: $(m \times k) \times (k \times n) = (m \times n)$.

Exam Practice 3: Inverses & Singularity

Question 3 (Past Exam)

If the determinant of a square matrix is zero ($\det(\mathbf{A}) = 0$), the matrix is:

- A) Invertible B) Singular (non-invertible) C) Symmetric D) Skew-symmetric

Exam Practice 3: Inverses & Singularity

Question 3 (Past Exam)

If the determinant of a square matrix is zero ($\det(\mathbf{A}) = 0$), the matrix is:

- A) Invertible B) Singular (non-invertible) C) Symmetric D) Skew-symmetric

Answer: B

$\det(\mathbf{A}) = 0$ means the matrix has linearly dependent columns and no inverse exists.

Exam Practice 4: Matrix Rank

Question 4 (Past Exam)

What is the rank of a matrix?

- A) The maximum number of linearly independent rows
- B) The maximum number of linearly independent columns
- C) The minimum number of rows
- D) Both A and B

Exam Practice 4: Matrix Rank

Question 4 (Past Exam)

What is the rank of a matrix?

- A) The maximum number of linearly independent rows
- B) The maximum number of linearly independent columns
- C) The minimum number of rows
- D) Both A and B

Answer: D

Row rank always equals column rank ($\text{rank}_{\text{row}}(\mathbf{A}) = \text{rank}_{\text{col}}(\mathbf{A})$).

Exam Practice 5: Determinant Formula

Question 5 (Past Exam)

What is the determinant of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

- A) $a + d$ B) $ad - bc$ C) $ac + bd$ D) $a^2 + d^2$

Exam Practice 5: Determinant Formula

Question 5 (Past Exam)

What is the determinant of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

- A) $a + d$ B) $ad - bc$ C) $ac + bd$ D) $a^2 + d^2$

Answer: B

By standard definition, $\det(\mathbf{A}) = ad - bc$.

Exam Practice 6: Identity Multiplication

Question 6 (Past Exam)

What is the result of multiplying any matrix by the identity matrix?

- A) The zero matrix
- B) The original matrix
- C) The transpose of the original matrix
- D) The inverse of the original matrix

Exam Practice 6: Identity Multiplication

Question 6 (Past Exam)

What is the result of multiplying any matrix by the identity matrix?

- A) The zero matrix
- B) The original matrix
- C) The transpose of the original matrix
- D) The inverse of the original matrix

Answer: B

$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. The identity matrix acts as scalar 1.

Lecture 1 Summary & Key Formulas

Core Matrix Formulas:

- Expected return: $\mathbb{E}[R_p] = \mathbf{w}'\mathbf{r}$
- Portfolio variance: $\sigma_p^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w}$
- 2×2 Inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Transpose of product: $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
- Inverse of product: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Key Concepts to Remember:

- Matrix multiplication is **not commutative** ($\mathbf{AB} \neq \mathbf{BA}$).
- System $\mathbf{Ax} = \mathbf{b}$ has unique solution $\iff \det(\mathbf{A}) \neq 0$.
- Arrow-Debreu state prices q_s price all assets under no-arbitrage.
- Covariance matrices must be **positive semi-definite** ($\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \geq 0$).
- Eigenvalues satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Next Lecture: Differential & Integral Calculus with Optimization