Financial Derivatives#15
Pricing American and Exotic Options via Numerical Methods

Andrianos E. Tsekrekos

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1 Preview

- Numerical methods for option pricing
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  - Formal treatment
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- The binomial asset pricing model
  - Assumptions and model set–up
  - No–arbitrage pricing and risk neutral valuation
  - Extension to multiple periods and convergence to Black and Scholes [2]
  - The effect of dividends and the pricing of American options

2 Numerical methods for option pricing

- The beauty and convenience of the Black and Scholes [2] framework is that it is "designed" in such a manner, that it produces an analytic solution (i.e. a closed–form equation) for the prices of European calls and puts.
- However, not all option contracts (e.g. American, Asian, Barrier) admit such convenient analytic solutions. In most such cases, the PDE that the price of a non–European option must satisfy does not admit a known solution. To price such options, several numerical methods have been proposed in the literature.

- In what follows, we will demonstrate the most popular numerical methods in the literature, initially in the context of European options, and then show how to extend them for non–European contracts.
- The most popular numerical methods are:
  - Monte Carlo simulation
  - Binomial asset pricing model
  - Finite difference methods

3 Monte Carlo simulation: Formal treatment

- In 1977 Boyle [3] demonstrated how European options can be priced via Monte Carlo simulation.
- The argument is based on the *risk–neutral valuation* result, i.e.

$$c = e^{-rT} E^{\mathbb{Q}} \left[\mathsf{max} \left(S_T - K \right)^+ \right]$$

$$p = e^{-rT} E^{\mathbb{Q}} \left[\mathsf{max} \left(K - S_T \right)^+ \right]$$

- So, if we can calculate the option's expected payoff in a risk–neutral world, the option price will be given by discounting it with the risk–free rate. Assuming that interest rates are constant, this basically involves *approximating the expected risk–neutral payoff by a large number of simulated possible asset paths*. This means the following steps:

– Divide the time horizon $[0, T]$ into N, equally spaced intervals, each of length Δt.

– For each time interval, sample M, normally distributed random numbers, with zero mean and unit standard deviation ($M \times N$ random numbers in total).

– Use the random number to produce $M \times N$ matrix of possible paths for the underlying asset in a risk–neutral world. For an asset S paying a continuous dividend yield and following a gBm [in a risk–neutral world, $\mathbb{Q}$],

$$dS = (r - \delta)\, S dt + \sigma S dz$$

$$S_{t+\Delta t} = S_t \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\,(\Delta t)^{\frac{1}{2}}\right] \tag{1}$$

N.B.: Equation (1) is slightly different from the discrete version we have used in our simulation example in lecture#6 (pp. 25–26). The above is the correct and exact discrete version of a gBm, while the one in #6 is only an approximation.

– At the option maturity, $T = N \times \Delta t$, calculate the option payoff for each of the M simulated paths (i.e. a $M \times 1$ vector)

$$\text{for a call, } (S_T - K)^+ \quad \text{or} \quad \text{for a put, } (K - S_T)^+$$

– Calculate the average payoff over all M paths, and then discount the result with e^{-rT}.

- Boyle [3] has shown that as $M \to +\infty$ and $N \to +\infty$ (i.e. $\Delta t \to 0$), the simulated option price converges to the correct, Black and Scholes price, $O_{MC} \to O_{B\&S}$.

4 Advantages and disadvantages

- Advantages of MC simulation method:
  - Can be used when the payoff of the contract depends on the actual path followed by the underlying variable S, as well as when it depends only on the final value of S.
  - Any stochastic process for S can be accommodated, as well as any payment patterns made by the underlying asset (e.g. dividends)
  - Can be extended to accommodate situations where the option from the derivative depends on several underlying market variables.
- Disadvantages of MC simulation method:
  - It is computationally time–consuming
  - Despite some recent research results (see for example, Longstaff and Schwartz [7]), it cannot easily handle situations where there are early exercise opportunities (i.e. American, Bermudan options).

5 Calculating the Greeks via Monte Carlo simulation

- The Greek letters covered in lecture #9–10 can be calculated via Monte Carlo simulation.
- Suppose that we are interested in the partial derivative of f with respect to x, where f is the value of the option contract and x is the value of an underlying asset or a parameter that affects the option value.
- To calculate $\frac{\partial f}{\partial x}$, use MC to produce one estimate of the option price, $\widehat{f}$ as we've seen. Then assume that parameter x increases slightly by Δx, and re–calculate a new value for the option $\widehat{f}^*$, the same way as $\widehat{f}$ (same number of time intervals N and random paths M).
- Then, an estimate of the Greek letter under review is given by

$$\frac{\widehat{f}^* - \widehat{f}}{\Delta x}$$

6 An exotic option example

- An *average–strike Asian put option* (denote Ap), gives its holder the right (without the corresponding obligation) to sell at time T and asset S for a price equal to the *average asset price realised* between time $t = 0$ and maturity time T, i.e.

$$Ap = \left(\frac{1}{i} \sum_{i=0}^{T} S_{t_i} - S_T \right)^{+}$$

- There is no closed–form solution for this option contract. However, we can use our Monte Carlo methodology to calculate its price.

- The only difference for this contract is that the strike price is different for each asset price path: for each of the M possible path, we need to take the average over time (i.e. over N path realisations). The result is the strike price that applies for this particular path.
- Once the above is repeated for all paths M, again calculate the option's payoff, take the average payoff, and then discount the result with e^{-rT} to get the option price.

7 The Binomial Asset Pricing Model

- In 1979, Cox, Ross and Rubinstein [4] published one of the most important pedagogical papers in option pricing. Their *binomial asset pricing model* goes as follows:
- Consider the following simple market: 2 assets, 2 possible states of nature
- Assets:
  1. Underlying asset
  2. Risk free asset
- States
  1. "u"— Price of the underlying goes up
  2. "d"— Price of the underlying goes down
- If market is complete and there is to be no–arbitrage, what should be the correct price of a European call option?

8 Underlying asset

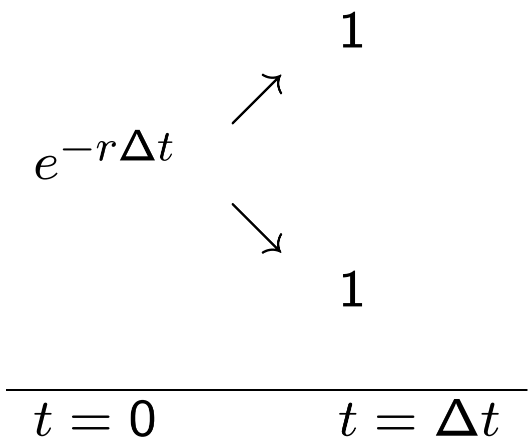
- The price of the underlying asset follows the *binomial process*:

$$
\begin{array}{ccc}
 & & u \cdot S_0 \\
 & \nearrow & \\
S_0 & & \\
 & \searrow & \\
 & & d \cdot S_0 \\
\hline
t = 0 & & t = \Delta t
\end{array}
$$

where S_0 is the price of the asset in the beginning of the period. The price at $t = \Delta t$ will be either $S_u = u \cdot S_0$ or $S_d = d \cdot S_0$.

9 Risk free asset

- The riskless asset pays £1 at $t = \Delta t$ no matter which state of nature prevails. Its price follows



- This asset is also referred to as *pure discount* or *zero–coupon bond* (with "normalised" face value of £1). If you buy a zero–coupon bond for $£e^{-r\Delta t}$ today, you will get $e^{-r\Delta t}e^{r\Delta t} = 1$ in a period's time.

- Assumption: constant, continuously compounded riskless rate.
- Parameters d, u and r verify the following relation:

$$0 < d < 1 < e^{r\Delta t} < u \tag{2}$$

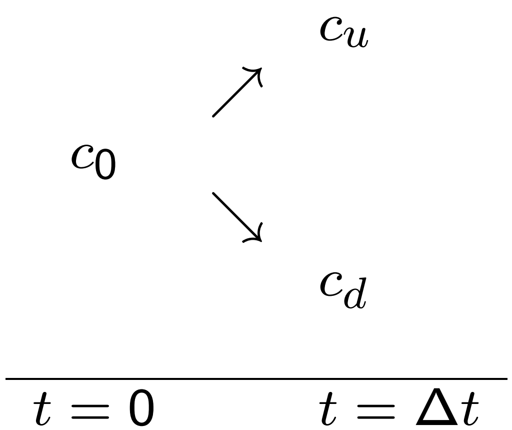
  - If $e^{r\Delta t} \geq u$ no one would invest in the stock since the rate of the riskless investment is at least as great, and sometimes greater than the return of the stock.
  - If $d \geq e^{r\Delta t}$ one could borrow money and invest in the stock since, even in the worst case, the stock price rises at least as fast as the debt used to buy it.
  - Sometimes one assumes $d = \frac{1}{u}$ to simplify results.

10 Isn't this too simple?

- In reality, stock price movements are far more complicated that those implied by the binomial asset pricing model.
- However, this simple model is useful because:
  1. Within the binomial model the concept of *no–arbitrage pricing* and its relation to *risk–neutral* pricing is clearly illuminated.
  2. The model is used widely in practice because with a sufficient number of time–steps, it provides a good, *computationally tractable* approximation to continuous–time models (which is the benchmark in the literature).
  3. The method can be extended to handle cases that the (continuous–time) B&S cannot.

11 Arbitrage pricing

- Depending on which state prevails at time $t = \Delta t$, a European call option will have different values denoted c_u and c_d



- To price the option, at time $t = 0$ we construct an *no–arbitrage portfolio* containing:
  - One underlying asset S_0
  - ϕ calls c_0
- The value of the portfolio is

$$V_0 = S_0 + \phi c_0$$

At $t = \Delta t$ we will have

$$\begin{array}{ccc} & & V_u = uS_0 + \phi c_u \\ & \nearrow & \\ V_0 & & \\ & \searrow & \\ & & V_d = dS_0 + \phi c_d \\ \hline t = 0 & & t = \Delta t \end{array}$$

- Now choose ϕ in such a manner, so that the no–arbitrage portfolio becomes *riskless*, i.e. it yields a risk free rate of return $e^{r\Delta t}$.
  Therefore, the no–arbitrage condition is:

$$V_u = V_d = V_0 e^{r\Delta t} \tag{3}$$

or equivalently

$$uS_0 + \phi c_u = dS_0 + \phi c_d = (S_0 + \phi c_0)\, e^{r\Delta t}$$

Basic algebra yields the solution

$$\phi = \frac{S_0\,(u-d)}{c_d - c_u} \overset{c_u \geq c_d}{\leq} 0 \tag{4}$$

and

$$c_0 = e^{-r\Delta t} \left\{ \frac{e^{r\Delta t} - d}{u-d} c_u + \left[1 - \frac{e^{r\Delta t} - d}{u-d} \right] c_d \right\} \tag{5}$$

- The coefficient ϕ is interpreted as *the number of call options necessary to include in an arbitrage portfolio containing one asset*.
- The negative sign of ϕ implies that we *sell* or *write* call options against the asset.
- In other words, ϕ is *the number of call options that one can write against the asset, so that a portfolio containing these and one asset is risk–free* (perfect hedge).
- The same analysis could be done if *we write one call and we want to know how many assets we must be long*. In this case, take the inverse of ϕ and reverse its sign. This is known as the *hedge ratio* or *"delta"* of a call.

$$\Delta = -\frac{1}{\phi} = \frac{c_u - c_d}{S_u - S_d} \geq 0 \qquad (6)$$

12 Risk–neutral probability

- In the no–arbitrage result for c_0 in (5), define

$$\mathbb{Q} \equiv \frac{e^{r\Delta t} - d}{u - d} \tag{7}$$

as the *risk–neutral* probability.

- Indeed, $\mathbb{Q}$ has the necessary properties of a probability measure. Property (2) ensures that

$$0 < \mathbb{Q} < 1$$

- The amazing thing about the pricing formula in (5) is that we do not need the *actual probabilities** of states "u" and "d", say $\mathbb{P}$ and $1 - \mathbb{P}$, to price options.

*Also known as: historical, objective, subjective, empirical or standard.

13 Binomial formula

- The price of a European option is equal to its expected value computed using risk–neutral probabilities $\mathbb{Q}$, $1-\mathbb{Q}$ and then discounted using the riskless rate r.
- From (5) using notation (7) we get:

$$c_0 = e^{-r\Delta t}\left[\mathbb{Q}\cdot c_u + (1-\mathbb{Q})\cdot c_d\right] = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\widetilde{c}_1|\mathcal{F}_0\right] \tag{8}$$

  where $\widetilde{c}_1$ denotes the random price of the option which could be either $c_1 = c_u$ or $c_1 = c_d$.
- The expectation taken using risk–neutral probabilities $\mathbb{Q}$, $1-\mathbb{Q}$ is denoted $\mathbb{E}^{\mathbb{Q}}[.]$ and the set $\{\mathbb{Q}, 1-\mathbb{Q}\}$ is called risk–neutral probability measure or $\mathbb{Q}$–measure. The expectation taken is conditional on the information $\mathcal{F}_0$ available at time $t=0$.

14 Multiperiod setting

- Let $t = 0, \Delta t, 2\Delta t, \ldots, n \cdot \Delta t, \ldots T$ where $T = N \cdot \Delta t$
- It is straightforward to generalise (8) to multiperiod binomial trees
- On each sub–tree

$$c_n = e^{-r\Delta t}\left[\mathbb{Q} c_{n+1}^{(u)} + (1-\mathbb{Q})\, c_{n+1}^{(d)}\right] = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\tilde{c}_{n+1} | \mathcal{F}_n\right] \qquad (9)$$

Use the above as many times as there are tree nodes between $t = 0$ and T (N times)

$$\begin{aligned} c_0 &= e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\tilde{c}_1 | \mathcal{F}_0\right] \\ &= e^{-r2\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\tilde{c}_2 | \mathcal{F}_1\right] | \mathcal{F}_0\right] \\ &= \ldots \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\ldots \mathbb{E}^{\mathbb{Q}}\left[\tilde{c}_T | \mathcal{F}_{T-\Delta t}\right] | \ldots | \mathcal{F}_1\right] | \mathcal{F}_0\right] \end{aligned}$$

- But from the properties of conditional expectations we know that[†]

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\widetilde{c}_{n+2}|\mathcal{F}_{n+1}\right]|\mathcal{F}_n\right] = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{c}_{n+2}|\mathcal{F}_n\right]$$

  Pursuing this further yields

$$c_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\widetilde{c}_T|\mathcal{F}_0\right]$$

  However, we know that at maturity, a call option is

$$\widetilde{c}_T = \left(\widetilde{S}_T - K\right)^+$$

[†]This property is referred to as the "tower law".

15 European option pricing formula

$$c_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\widetilde{S}_T - K\right)^+ |\mathcal{F}_0\right] \tag{10}$$

The value of a European call is a (conditional) expectation — under the risk–neutral probability measure—of its payoff at maturity, discounted using the riskless interest rate. This is the exact same result as equation (11) in lecture #4.

16 A two–period put option example

You observe a European put written on a stock with $S_0 = £50$, $K = £52$, $r = 5\%$ p.a. and $T = 2$ years. Let $\Delta t = 1$ year, $u = 1.20$ and $d = 0.80$ so that

$$
\begin{array}{ccccc}
 & & & & S_{uu} = 72 \\
 & & & \nearrow & \\
 & & S_u = 60 & & \\
 & \nearrow & & \searrow & \\
S_0 = 50 & & & & S_{ud} = S_{du} = 48 \\
 & \searrow & & \nearrow & \\
 & & S_d = 40 & & \\
 & & & \searrow & \\
 & & & & S_{dd} = 32 \\
 & & & & \\
t = 0 & & t = 1 & & T = 2 \text{ years}
\end{array}
$$

Use a binomial tree to price the option.

$$
\begin{array}{ccccl}
 & & & & p_{uu} = (K - S_{uu})^+ = (52 - 72)^+ = 0 \\
 & & & \nearrow & \\
 & & p_u & & \\
 & \nearrow & & \searrow & \\
p & & & & p_{ud} = p_{du} = (K - S_{ud})^+ = (52 - 48)^+ = 4 \\
 & \searrow & & \nearrow & \\
 & & p_d & & \\
 & & & \searrow & \\
 & & & & p_{dd} = (K - S_{dd})^+ = (52 - 32)^+ = 20 \\
t = 0 & & t = 1 & & T = 2 \text{ years}
\end{array}
$$

SOLUTION:

1. Calculate the payoff at all final tree nodes $(T = 2)$.
2. Calculate the risk–neutral probabilities $\mathbb{Q} = \frac{e^{0.05\times1}-0.8}{1.2-0.8} = 0.6282$ and $1 - \mathbb{Q} = 0.3718$
3. Use equation (10): Multiply the payoff of each ending node with the risk–neutral probability of reaching this node, discount with the risk–free rate to time $t = 0$ and sum

$$p = e^{-0.05\times2}\left[0.6282^2 \times 0 + 2 \times 0.6282 \times 0.3718 \times 4 + 0.3718^2 \times 20\right]$$
$$= 4.1923$$

17 From the Binomial model to Black–Scholes

- We will demonstrate how the Binomial model can be extended to continuous–time and how it relates to the Black and Scholes [2] formula
- A binomial tree with N steps (each of size Δt, such that $T = N \cdot \Delta t$) has $N + 1$ end–nodes and 2^N possible paths of reaching the end–nodes.
- The risk–neutral probability of reaching a particular end–note is given by

$$\mathcal{B}(n|N, \mathbb{Q}) = \frac{N!}{(N-n)!n!} \mathbb{Q}^n (1 - \mathbb{Q})^{N-n}$$

while the call option payoff is

$$\left(u^n d^{N-n} S_0 - K\right)^+$$

where $n = 0, 1, 2, \ldots, N$ is the number of upwards movements in the underlying asset.[‡] Thus

$$c = e^{-rT} \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} \mathbb{Q}^n (1 - \mathbb{Q})^{N-n} \left(u^n d^{N-n} S_0 - K \right)^+$$

- Separate the two elements $u^n d^{N-n} S_0$ and K in the max operator by only considering the paths that end up with a positive payoff

$$c = S_0 \sum_{n=a}^{N} \frac{N!}{(N-n)!n!} \left(\mathbb{Q}' \right)^n \left(1 - \mathbb{Q}' \right)^{N-n} - K e^{-rT} \sum_{n=a}^{N} \frac{N!}{(N-n)!n!} \mathbb{Q}^n (1 - \mathbb{Q})^{N-n}$$

[‡]Remember, $\forall x \in \mathbb{Q}$, $x! = 1 \cdot 2 \cdot \ldots \cdot x$, and $0! \equiv 1$.

$$c = S_0 \mathcal{B}\left(n \geq a|N, \mathbb{Q}'\right) - Ke^{-rT}\mathcal{B}\left(n \geq a|N, \mathbb{Q}\right) \tag{11}$$

where $\mathcal{B}\left(n \geq a|N, \mathbb{Q}'\right)$ is the complementary binomial distribution function, a is the smallest non–negative integer greater than

$$a \overset{\in\mathbb{Q}}{>} \frac{\ln\left(K/S_0 d^n\right)}{\ln\left(u/d\right)}$$

and

$$\mathbb{Q}' = \frac{u}{e^{rT}}\mathbb{Q}$$

- Cox, Ross and Rubinstein [4] show that if the u and d parameters are set so as to match the volatility σ of the Black–Scholes framework, i.e.

$$u = e^{\sigma (T/N)^{0.5}}$$
$$d = e^{-\sigma (T/N)^{0.5}}$$

then equation (11) converges to the Black and Scholes [2] for a large number of time steps N since

$$\lim_{N\to+\infty} \mathcal{B}\left(n \geq a|N, \mathbb{Q}'\right) = \Phi\left(d_1\right)$$

$$\lim_{N\to+\infty} \mathcal{B}\left(n \geq a|N, \mathbb{Q}\right) = \Phi\left(d_2\right)$$

18 The effect of a known dividend yield

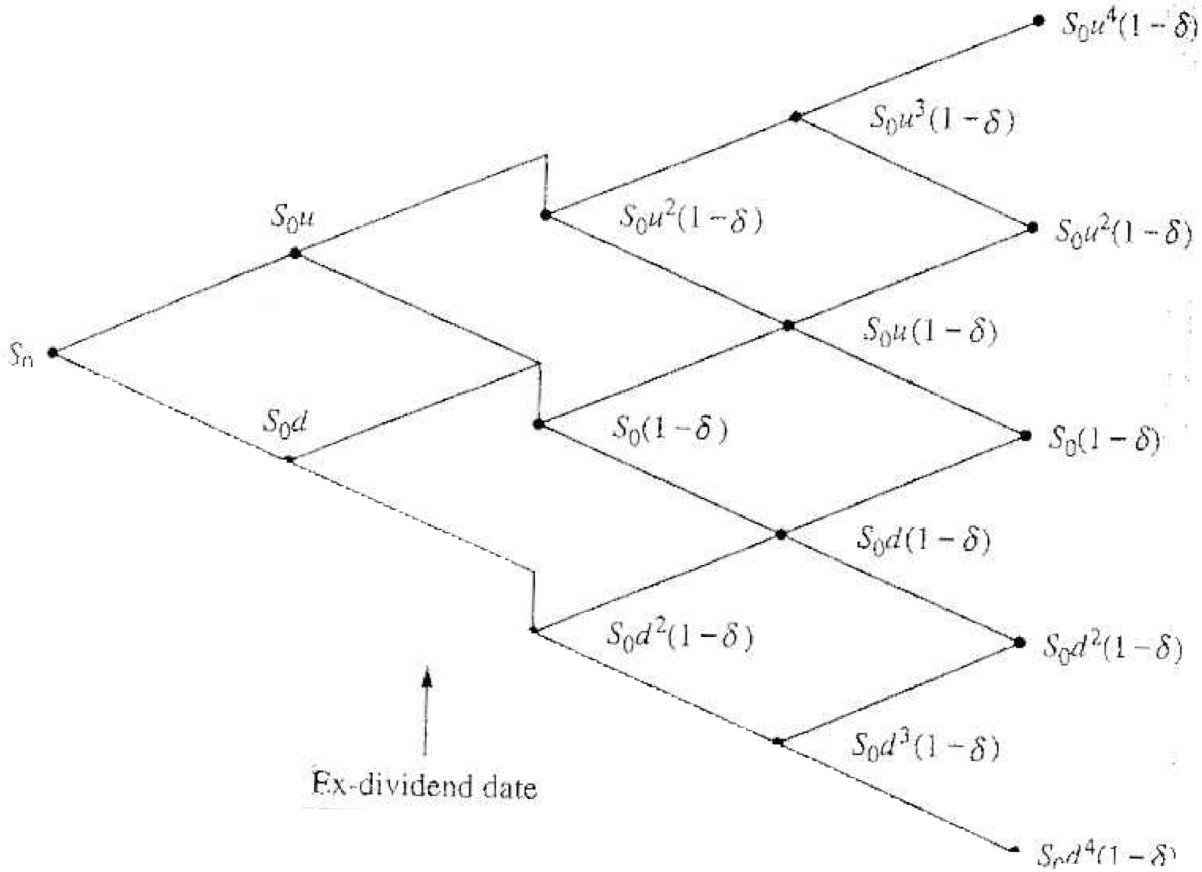
- In cases where the underlying asset pays a proportional dividend yield δ, the stock prices on tree nodes would be

$$S_{\underbrace{u\ldots u}_{j}\underbrace{d\ldots d}_{i-j}} = \begin{cases} S_0 u^j d^{i-j} & \text{for } i\Delta t \text{ before ex–div date} \\ \\ S_0\,(1-\delta)\,u^j d^{i-j} & \text{for } i\Delta t \text{ after ex–div date} \end{cases}$$

with $i = 0, 1, \ldots N$ and $j = 0, 1, \ldots i$.

- The risk–neutral probability $\mathbb{Q}$ in equation (7) is accordingly modified

$$\mathbb{Q} \equiv \frac{e^{(r-\delta)\Delta t} - d}{u - d} \tag{12}$$



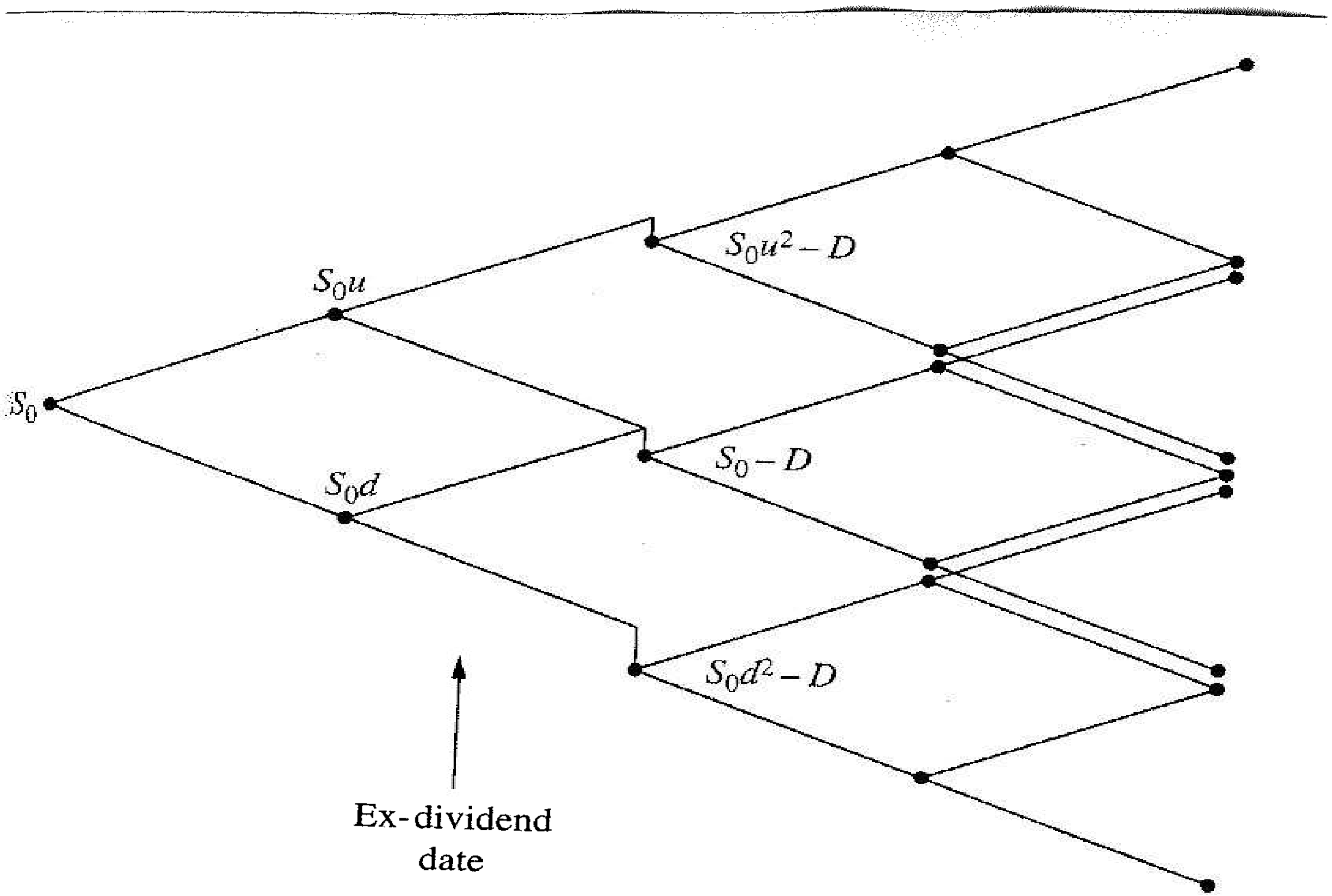
$S_0u^4(1-\delta)$
$S_0u^3(1-\delta)$
S_0u
$S_0u^2(1-\delta)$
$S_0u^2(1-\delta)$
$S_0u(1-\delta)$
S_0
S_0d
$S_0(1-\delta)$
$S_0(1-\delta)$
$S_0d(1-\delta)$
$S_0d^2(1-\delta)$
$S_0d^2(1-\delta)$
$S_0d^3(1-\delta)$
Ex-dividend date

19 The effect of a known dollar dividend

- If the dividend is known as a dollar amount and not as a proportion of the stock price, then the resulting binomial tree is *non-recombining*, i.e. after the ex–div date the tree has $2i$ instead of $i+1$ nodes. This increases the computational complexity considerably.
- In this case, the stock prices on tree nodes would be

$$S_{\underbrace{u\ldots u}_{j}\underbrace{d\ldots d}_{i-j}} = \begin{cases} S_0 u^j d^{i-j} & \text{for } i\Delta t \text{ before ex–div date} \\ \left(S_0 u^q d^{q-k} - D\right) u^{i-k} j^{i-j-k} & \text{for } i\Delta t \text{ after ex–div date} \end{cases}$$

with $i = 0, 1, \ldots N$, $j = 0, 1, \ldots i$, $q = 0, 1, \ldots k$ and k the lowest integer greater than the ex–div date.



S_0
S_0u
S_0d
$S_0u^2 - D$
$S_0 - D$
$S_0d^2 - D$
Ex-dividend
date

20 Pricing American options in the binomial model

- An American–style option, unlike a European one, gives the holder the right (without the corresponding obligation) to buy (call, C) or sell (put, P) the underlying asset for a specific price, *at any point in time up until the specified maturity date*.
- In cases where the underlying asset pays no cash flows (e.g. dividends), we have shown (see lecture #3, p.11) that an American call will never be optimally early–exercised. However, the American put may be early exercised in such cases. How can we price an American put on a non–dividend paying asset?

- Firstly, we need to modify our backward–induction equation (9)

$$P_n = \max\left\{\underbrace{K - S_n}_{\text{immediate}}, \underbrace{e^{-r\Delta t}\left[\mathbb{Q}P^{(u)}_{n+1} + (1-\mathbb{Q})\,P^{(d)}_{n+1}\right]}_{\text{put ``alive''}}\right\} \qquad (13)$$

- Again we would start from the option maturity $T = N \cdot \Delta t$, where

$$P_N = \left(K - \widetilde{S}_N\right)^+$$

and work backwards to $t = 0$ using equation (13) as in the European case.

21 Two–period case

Let $N = 2$. At *maturity*:

$$\widetilde{P}_2 = \left(K - \widetilde{S}_2\right)^+ \tag{14}$$

where $S_2 \in \left\{S_2^{(uu)}, S_2^{(ud)}, S_2^{(dd)}\right\}$. At the *intermediate node*:

$$\widetilde{P}_1 = \max\left\{K - \widetilde{S}_1, e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\widetilde{P}_2|\mathcal{F}_1\right]\right\} \tag{15}$$

where $S_1 \in \left\{S_1^{(u)}, S_1^{(d)}\right\}$. At the *onset*:

$$P_0 = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\widetilde{P}_1|\mathcal{F}_0\right] \tag{16}$$

where $P_1 \in \left\{P_1^{(u)}, P_1^{(d)}\right\}$.

- Combine (14), (15) and (16) to get

$$P_0 = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\underbrace{\max\left\{\left(K-\widetilde{S}_1\right)^+, e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\underbrace{\left(K-\widetilde{S}_2\right)^+}_{\widetilde{P}_2}\middle|\mathcal{F}_1\right]\right\}}_{\widetilde{P}_1}\middle|\mathcal{F}_0\right]$$

- Thus to calculate the put price P_0, we are interested only in the (conditional) expected present values of $\left(K-\widetilde{S}_2\right)^+$ and $\left(K-\widetilde{S}_1\right)^+$. To compute that we need to consider all possible "paths" of the asset price, i.e. couplets $\{S_1, S_2\}$.

- If the option is to be exercised, it will be:
  - *Either* at $t = \Delta t$ with payoff $\left(K - \widetilde{S}_1\right)^+ > 0$
  - *Or* at $t = 2\Delta t$ with payoff $\left(K - \widetilde{S}_2\right)^+ > 0$
    and the random variable $\tau \in \{\Delta t, 2\Delta t\}$ is called a "stopping time".

22 Numerical examples

- Consider the following set of parameters:[§]

The price of the underlying asset at $t = 0$	$S_0 = 300$
The strike price	$K = 300$
"up" factor	$u = 2$
"down" factor	$d = \frac{1}{u} = \frac{1}{2}$
time step	$\Delta t = 1$
Discount factor	$e^{-r\Delta t} = \frac{4}{5}$

[§]Parameters values are rather unrealistic; however they should help demonstrate the principles.

23 Preliminaries

- Build the underlying price "tree" and calculate the risk–neutral probabilities.

$$
\begin{array}{ccccc}
 & & & & S_{uu} = 1200 \\
 & & & \nearrow & \\
 & & S_u = 600 & & \\
 & \nearrow & & \searrow & \\
S_0 = 300 & & & & S_{ud} = S_{du} = 300 \\
 & \searrow & & \nearrow & \\
 & & S_d = 150 & & \\
 & & & \searrow & \\
 & & & & S_{dd} = 75 \\
t = 0 & & t = 1 & & t = T = 2
\end{array}
$$

$$\mathbb{Q} \equiv \frac{e^{r\Delta t} - d}{u - d} = \frac{5/4 - 1/2}{2 - 1/2} = \frac{1}{2} \qquad 1 - \mathbb{Q} = 1 - \frac{1}{2} = \frac{1}{2}$$

24 European call

$$
\begin{array}{ccccc}
 & & & & 900 = (1200 - 300)^{+} = c_{uu} \\
 & & & \swarrow & \\
 & & 360 & & \\
 & \swarrow & & \nwarrow & \\
c_0 = 144 & & & & 0 = (300 - 300)^{+} = c_{ud} \\
 & \nwarrow & & \swarrow & \\
 & & 0 & & \\
 & & & \nwarrow & \\
 & & & & 0 = (75 - 300)^{+} = c_{dd}
\end{array}
$$

$$c_u = e^{-r\Delta t}\left[\mathbb{Q}c_{uu} + (1 - \mathbb{Q})\,c_{ud}\right] = 360$$

$$c_0 = e^{-r\Delta t}\left[\mathbb{Q}c_u + (1 - \mathbb{Q})\,c_d\right] = 144$$

25 American call

$$
\begin{array}{ccccc}
 & & & & 900 = (1200 - 300)^{+} = C_{uu} \\
 & & & \swarrow & \\
 & & 360 = \max\left\{ \underbrace{60}_{S_u - K}, 360 \right\} & & \\
 & \swarrow & & \nwarrow & \\
C_0 = 144 & & & & 0 = (300 - 300)^{+} = C_{ud} \\
 & \nwarrow & & \swarrow & \\
 & & 0 = \max\left\{ \underbrace{-150}_{S_u - K}, 0 \right\} & & \\
 & & & \nwarrow & \\
 & & & & 0 = (75 - 300)^{+} = C_{dd}
\end{array}
$$

26 European put

$$
\begin{array}{ccccc}
 & & & & 0 = (300 - 1200)^{+} = p_{uu} \\
 & & & \swarrow & \\
 & & 0 & & \\
 & \swarrow & & \nwarrow & \\
p_0 = 36 & & & & 0 = (300 - 300)^{+} = p_{ud} \\
 & \nwarrow & & \swarrow & \\
 & & 90 & & \\
 & & & \nwarrow & \\
 & & & & 225 = (300 - 75)^{+} = p_{dd}
\end{array}
$$

$$p_d = e^{-r\Delta t}\left[\mathbb{Q}p_{uu} + (1 - \mathbb{Q})\, p_{ud}\right] = 90$$
$$p_0 = e^{-r\Delta t}\left[\mathbb{Q}p_u + (1 - \mathbb{Q})\, p_d\right] = 36$$

27 American put

$$0 = (300 - 1200)^{+} = P_{uu}$$

$$\swarrow$$

$$0 = \max\left\{\underbrace{-300}_{K-S_u}, 0\right\}$$

$$\swarrow \qquad \nwarrow$$

$$P_0 = 60 \qquad\qquad 0 = (300 - 300)^{+} = P_{ud}$$

$$\nwarrow \qquad \swarrow$$

$$150 = \max\left\{\underbrace{150}_{K-S_u}, 90\right\}$$

$$\nwarrow$$

$$225 = (75 - 300)^{+} = P_{dd}$$

- Note that

$$\underbrace{P_0}_{\text{American}} = 60 > 36 = \underbrace{p_0}_{\text{European}}$$

$$\underbrace{C_0}_{\text{American}} = 144 = 144 = \underbrace{c_0}_{\text{European}}$$

28 Advantages and disadvantages

- Advantages of the binomial model:
  - Can easily handle situations where there are early exercise opportunities.
  - Can be used when the payoff of the contract depends on the actual path followed by the underlying variable S, as well as when it depends only on the final value of S.
  - Demonstrates clearly and in an easy framework the concepts of no–arbitrage pricing and risk neutral valuation.
- Disadvantages of the binomial model:
  - It is impossible to extend the model in situations where the option from the derivative depends on more than two underlying market variables.
  - Not all stochastic processes for the underlying asset S can be accommodated.

29 Reading

- Hull [5], Chapters 10, 18 (pp. 392–414), 19–20 (advanced reading)
- Baxter and Rennie [1], Chapter 2
- Jarrow and Turnbull [6], Chapters 5, 17, 19, 20
- Stulz [8], chapter 12
- Cox, Ross and Rubinstein [4]

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