Financial Derivatives
Chapter #12
Introduction to Stochastic Calculus

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1 Preview

- Modelling asset prices and interest rates as stochastic processes.
- Markov, Wiener, generalised Wiener and Itô processes.
- Geometric Brownian motion and process for interest rates.

2 Modelling asset prices

- The price and payoff of a derivative contract depends not only on the current price of the underlying asset (S_0 for futures/forwards, r_0 or L_t for FRAs, T–bill futures, etc.), but also on the future (and currently unknown) asset price ($\widetilde{S}_T$, $\widetilde{r}_{T,T+m}$, $L_{T,T+m}$) at maturity or delivery date, T.
- To make things even more complicated, the pricing of some derivative contracts, e.g. American–style options, depends on all possible asset prices up to and including the maturity date T ($\widetilde{S}_\tau$, $t \leq \tau \leq T$).
- Thus, the first step towards analysing and pricing derivatives is a concrete modelling framework for (underlying future) asset prices or interest rates.

Asset prices or interest rates in the financial engineering literature are assumed to follow *continuous–time Markov stochastic processes*. What is all that?

3 Stochastic processes

- Any variable whose value changes over time in an uncertain way is said to follow a stochastic process.
- Stochastic processes can be categorised as discrete–time or continuous–time:
  - the value of a variable that follows a *discrete–time* stochastic process can change only at certain fixed points in time
  - the value of a variable that follows a *continuous–time* stochastic process can change at any time

- Stochastic processes can also be categorised as discrete–variable or continuous–variable:
  - in a *continuous–variable* stochastic process, the variable can take any value within a certain range
  - in a *discrete–variable* stochastic process, only certain values are possible

	Discrete–time	Continuous–time
Discrete–variable	No. of goals scored by M. Owen	Accidents on M1
Continuous–variable	Magnitude of annual rainfall	Temperature

- Continuous–variable, continuous–time stochastic processes are used in finance to model asset prices or rates.

4 Markov processes

- A variable assumed to follow a *Markov* stochastic process* has the property that only the current value of the variable is relevant for predicting the future.
- The past history of the variable and the "path" followed by the variable before the current value has emerged are irrelevant. That is why non–Markov processes are known as path–dependent.
- Modelling financial prices as Markov stochastic processes implies that we cannot tell anything about the asset's price "tomorrow" by using its price history up to today. This is consistent with the weak–form of the market efficiency hypothesis.

*Andrei Andreyevich Markov, Russian mathematician (born 1856, died 1922).

5 Some statistics

- Consider a variable z following a Markov process

$$\begin{aligned} z_t &= z_0 + \sum_{i=1}^{t} \varepsilon_i \\ \varepsilon &\sim \text{i.i.d. } \mathcal{N}(0,1) \end{aligned} \tag{1}$$

This process is known as a *random walk*. Verify that

$$z_t - z_0 = \sum_{i=1}^{t} \varepsilon_i \tag{2}$$

and thus

$$(z_1 - z_0) \sim \mathcal{N}(0,1)$$

$$\begin{aligned}(z_2 - z_0) &\sim \mathcal{N}\left(0, 2^{1/2}\right)\\ &\vdots \\ (z_T - z_0) &\sim \mathcal{N}\left(0, T^{1/2}\right)\end{aligned}$$

since adding i.i.d. normally distributed ε's results in a normal distribution in which the mean is the sum of the means, the *variance* is the sum of the variances (and thus the *standard deviation* is the *square root* of the sum of the standard deviations).

6 Wiener, generalised Wiener and Itô processes

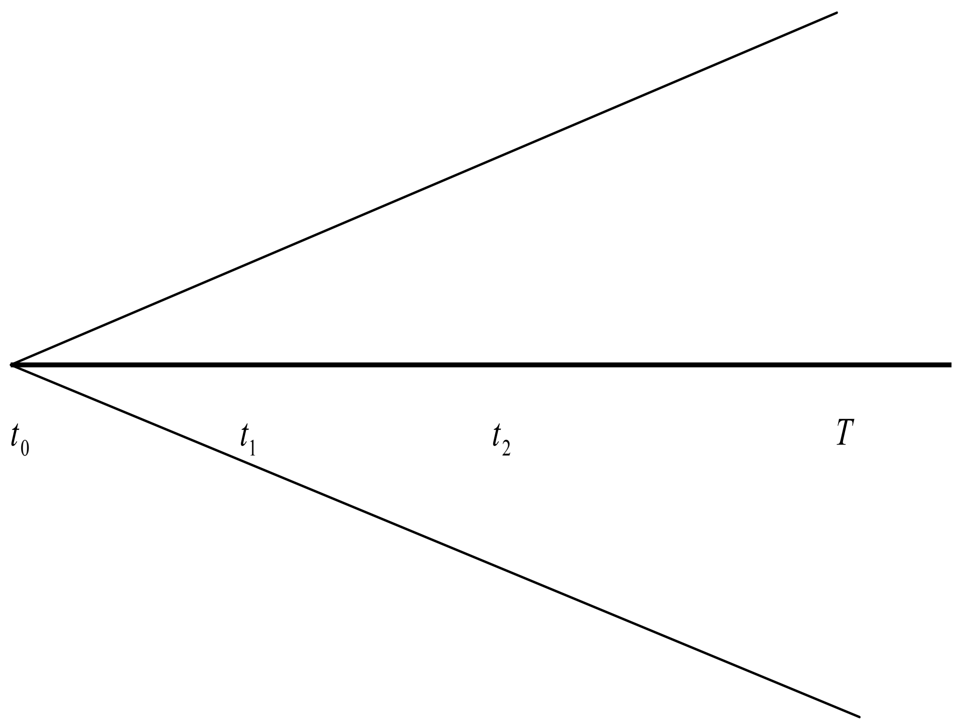
- The process in equation (2) is known as a *Wiener* process.† It had been used extensively in physics, where it is referred to as *Brownian motion*.‡ Writing equation (2) as

$$\begin{aligned} z(T) - z(0) &= \sum_{i=1}^{N} \varepsilon_i (\Delta t)^{\frac{1}{2}} \\ N &= \frac{T}{\Delta t} \end{aligned} \tag{3}$$

  and taking the limit as $\Delta t \to 0$ yields the Brownian motion as the "infinitesimal time–step" limit of the discrete–time random walk.

†Norbert Wiener, Russian mathematician (born 1894, died 1964)

‡Robert Brown, English botanist (born 1773, died 1858)


t_0
t_1
t_2
T

- Formally, the properties of the Brownian motion are:
  - The change dz during a small period of time dt is

$$dz = \varepsilon (dt)^{\frac{1}{2}} \tag{4}$$

$$\varepsilon \sim \mathcal{N}(0,1) \tag{5}$$

  - The values dz for any two different short time intervals dt are independent
- These two properties imply

$$\text{mean of } dz = 0 \qquad \text{mean of } [z(T) - z(0)] = 0$$

$$\text{variance of } dz = dt \qquad \text{variance of } [z(T) - z(0)] = T$$

$$\text{st. deviation of } dz = (dt)^{\frac{1}{2}} \qquad \text{st. deviation of } [z(T) - z(0)] = (T)^{\frac{1}{2}}$$

- If the random term ε in (5) is allowed to follow a general normal distribution with mean μ and standard deviation σ, $\mathcal{N}(\mu, \sigma)$, then a variable x is said to follow a *generalised Wiener* process (or *arithmetic Brownian motion*—aBm)

$$dx = \underbrace{\mu dt}_{\text{drift term}} + \underbrace{\sigma dz}_{\text{volatility term}} \quad (6)$$

$$= \mu dt + \sigma\varepsilon (dt)^{\frac{1}{2}} \quad (7)$$

- This implies

$$\text{mean of } dx = \mu dt \qquad \text{mean of } [x(T) - x(0)] = \mu T$$

$$\text{variance of } dx = \sigma^2 dt \qquad \text{variance of } [x(T) - x(0)] = \sigma^2 T$$

$$\text{st. deviation of } dx = \sigma (dt)^{\frac{1}{2}} \qquad \text{st. deviation of } [x(T) - x(0)] = \sigma (T)^{\frac{1}{2}}$$

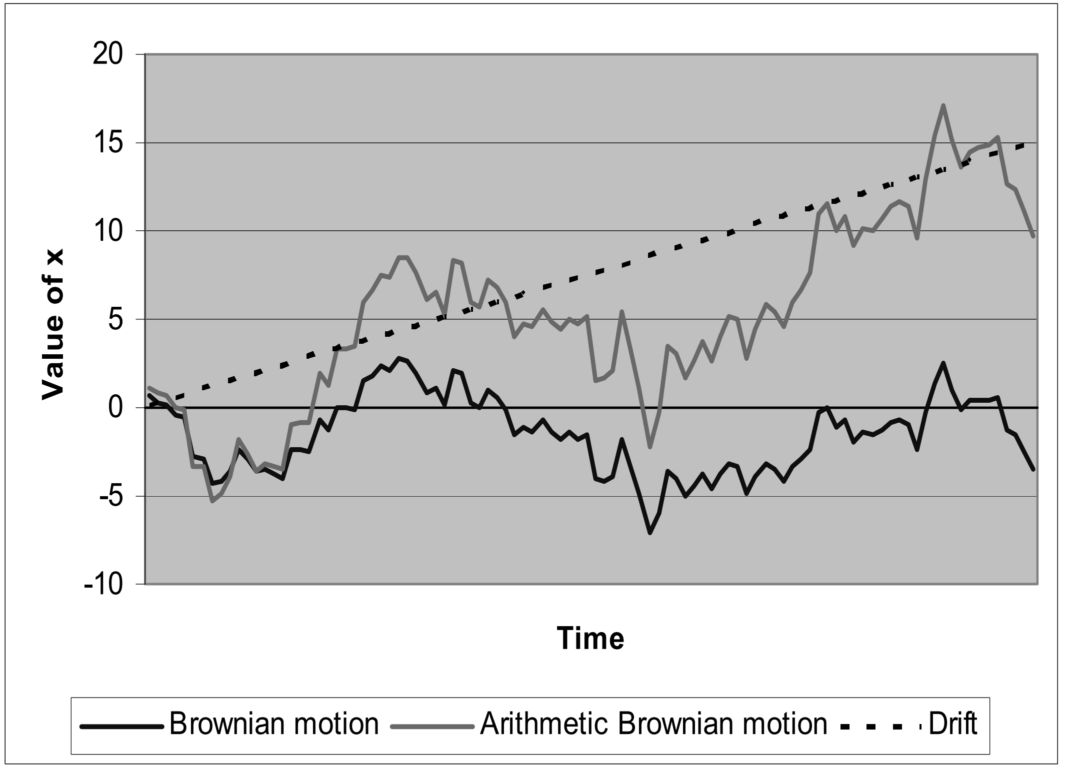
- More generally, any generalised Wiener process with drift and volatility terms that can be functions of the underlying variable x and time t,

$$dx = \alpha\left(x, t\right) dt + b\left(x, t\right) dz$$

are known as *Itô processes*.

- The Brownian motion (equation (3)) and the arithmetic Brownian motion (equation (6)) are of course Itô processes. So are the following examples

$$\begin{aligned} dx &= x dt + \sigma x^{\frac{3}{2}} dz \\ dx &= \sigma dz \\ dx &= \lambda x t^2 dt + \nu t^2 dz \end{aligned}$$


20
15
10
5
0
-5
-10
Value of x
Time
Brownian motion
Arithmetic Brownian motion
Drift

7 Modelling financial asset prices for derivative pricing

- The first financial engineering application was the option pricing model of Bachelier§ [1] in 1900.
- In his Ph.D. thesis, Bachelier derived a formula for the price of the options that were traded then in the Parisian Bourse¶ under the assumption that underlying asset prices follow arithmetic Brownian motions.
- Despite the direct link between option pricing theory and Bachelier's work, his economic rational is flawed from today's point of view. His assumption that asset prices can follow an aBm is unreasonable due to limited liability.
- Asset prices cannot go negative. For example, share prices, even when firms go bankrupt, they become worthless (i.e. zero price) and do not require shareholders to make extra payments.

§Louis Bachelier, French mathematician (born 1870, died 1946).

¶what are now called barrier options.

- However, the arithmetic Brownian motion (equation (6)) has the undesirable property that regardless of the initial value x_0 and the drift/volatility parameter values, the probability that x will become negative in the future is *non–zero*.

8 Geometric Brownian motion

- Samulelson [3], who rediscovered Bachelier's work about 50 years later, identified and modified the erroneous asset price assumption by proposing an alternative stochastic process

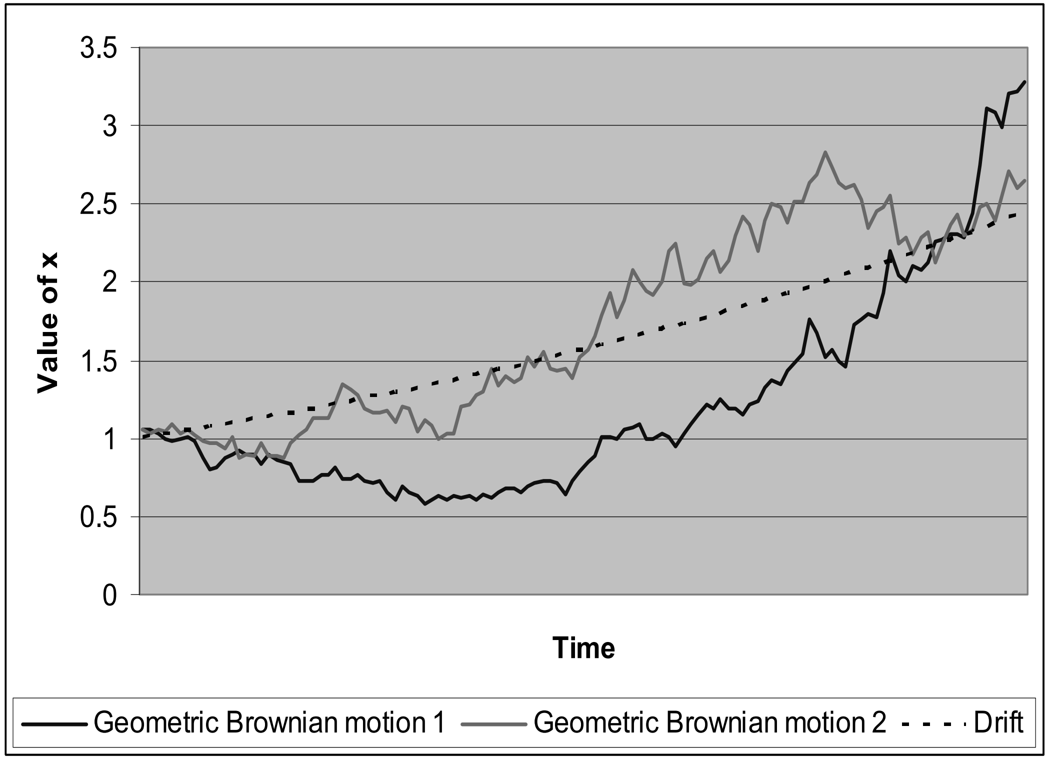
$$\begin{aligned} dx &= \mu x dt + \sigma x dz \\ \frac{dx}{x} &= \mu dt + \sigma dz \end{aligned} \tag{8}$$

  known as a *geometric Brownian motion* (gBm).
- A gBm belongs to the family of Itô processes and is also a continuous–time Markov stochastic process.
- Moreover, a gBm has the desirable property that $x > 0$ always, as long as $x_0 > 0$.

- Equation (8) implies that stock prices are *lognormally* distributed, or alternatively that stock price *returns are normally* distributed

$$\frac{\Delta x}{x} \sim \mathcal{N}\left(\mu \Delta t, \sigma \left(\Delta t\right)^{\frac{1}{2}}\right)$$

- Black and Scholes [2] used the process in (8) to model share prices in their Nobel prize–winning option pricing formula. Since their work, a gBm is the usual assumption for asset prices in finance.


3.5
3
2.5
2
1.5
1
0.5
0
Value of x
Time
Geometric Brownian motion 1
Geometric Brownian motion 2
Drift

9 Should we use a gBm for interest rates?

- Despite the fact that asset prices are usually assumed to follow gBm processes in the financial engineering literature, whether this process is appropriate for modelling the yield curve of the interest rates is questionable.
- One important difference between interest rates and stock prices is that while one expects stock prices to appreciate over a long period of time, time series of interest rates appear to be pulled back to some long–run average level over time.
- This phenomenon is known as *mean reversion*. When interest rates are high (low), mean reversion tends to cause them to have a negative (positive) drift.
- There is compelling economic arguments in favour of mean reversion. When interest rates are high, the economy tends to slow down and there is low

demand for funds from borrowers. As a result, rates decline. An inverse argument applies when rates are low.

- The simplest mean–reverting process—also known as an *Ornstein–Uhlenbeck process*—is the following:

$$\begin{aligned} dx &= \eta\left(\overline{x} - x\right) dt + \sigma dz \\ dx &= \left(\alpha + \beta x\right) dt + \sigma dz \end{aligned}$$

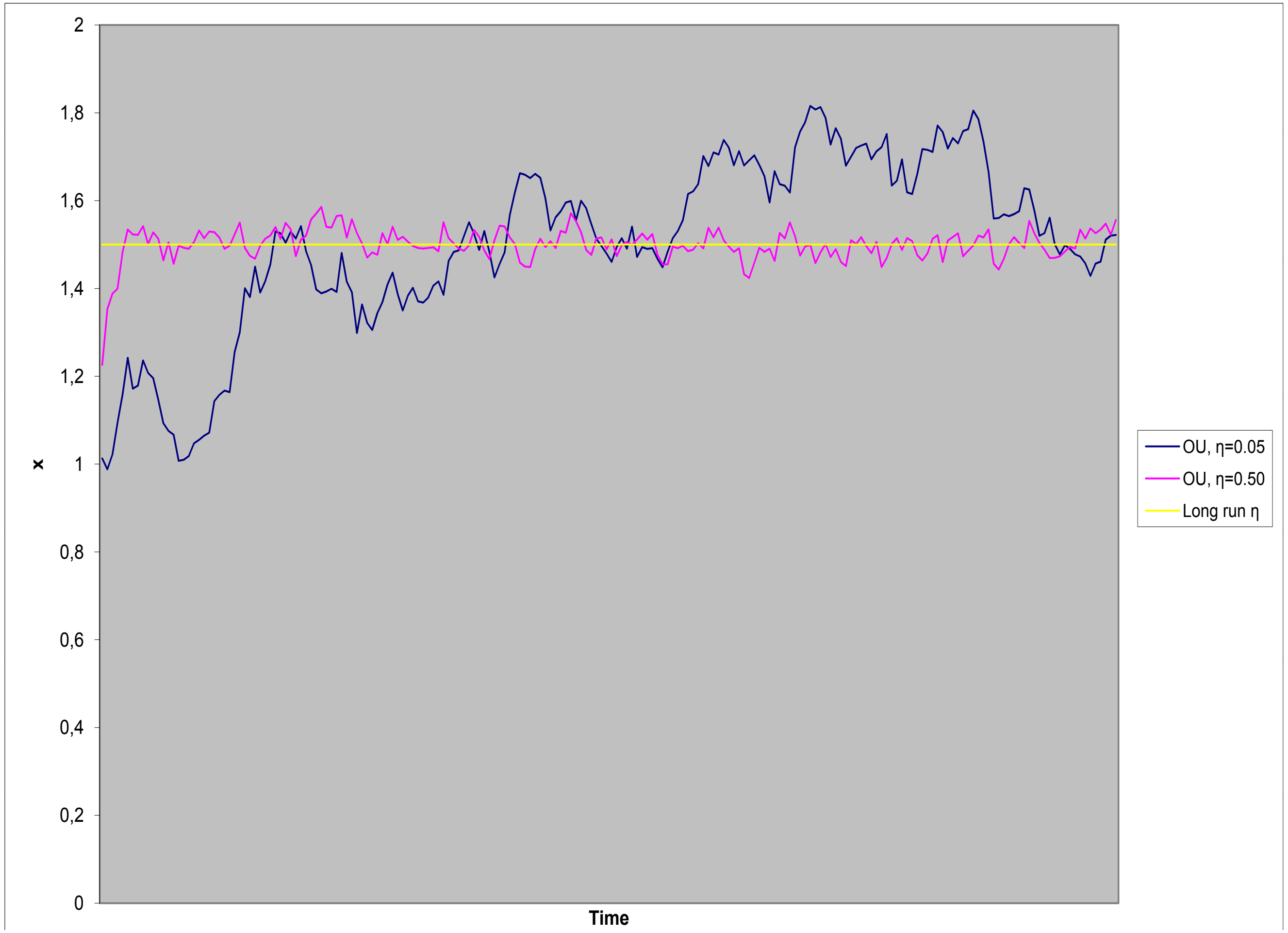
where η is the speed of reversion, and $\overline{x}$ is the "normal" or "long–run" level of x, i.e. the level at which x tends to revert to. Sometimes the second formulation is used where

$$\begin{aligned} \beta &= -\eta \\ \alpha &= \frac{\overline{x}}{\eta} \end{aligned}$$

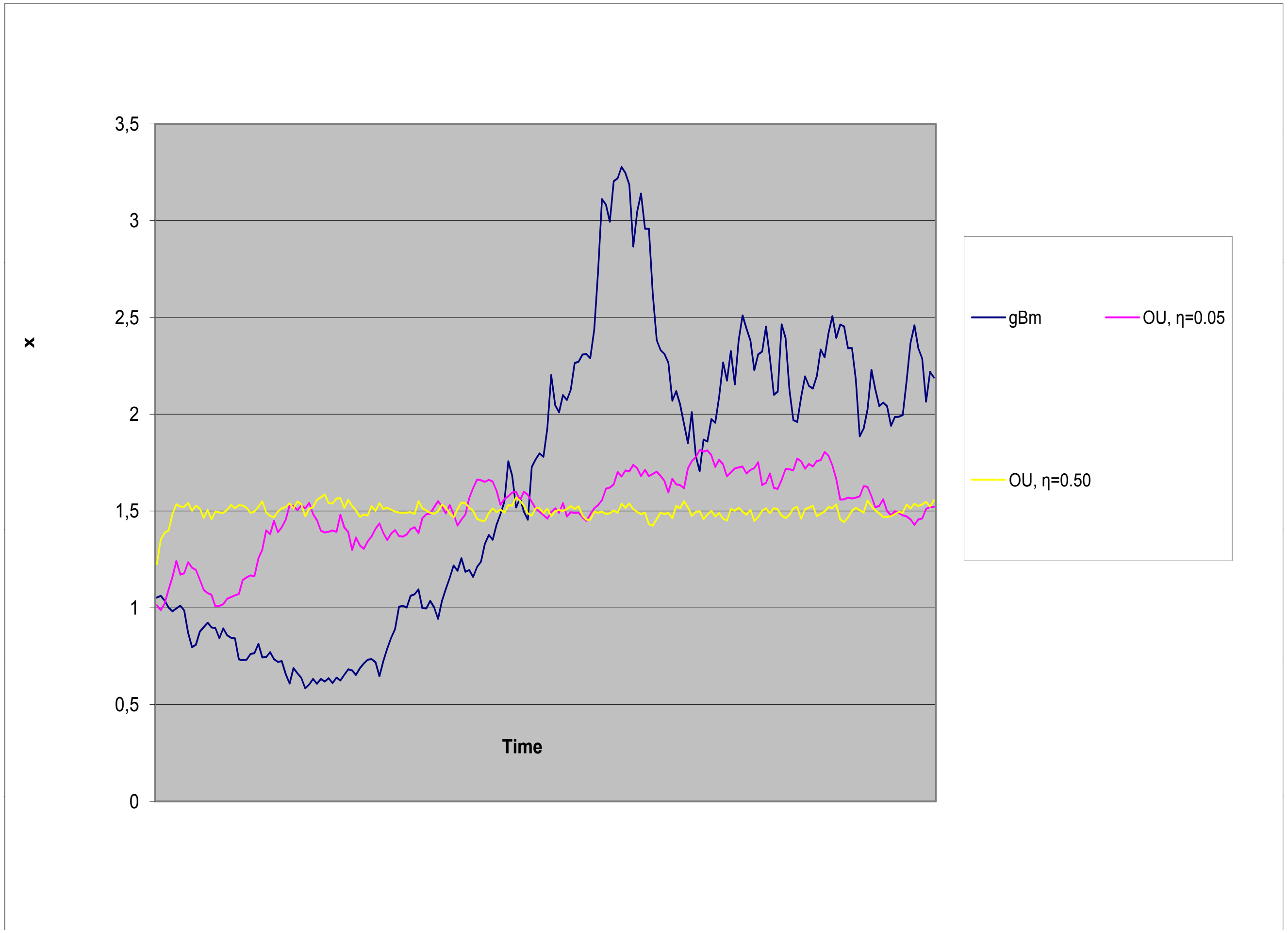
- The Ornstein–Uhlenbeck mean–reverting process belongs to the family of Itô processes and is also a continuous–time Markov stochastic process. It implies that

$$\text{mean of } x(T) = \overline{x} + [x(0) - \overline{x}]\, e^{-\eta T}$$

$$\text{variance of } x(T) - \overline{x} = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta T}\right)$$



2
1,8
1,6
1,4
1,2
1
0,8
0,6
0,4
0,2
0
x
Time
OU, η=0.05
OU, η=0.50
Long run η


3,5
3
2,5
2
1,5
1
0,5
0
x
Time
gBm
OU, η=0.05
OU, η=0.50

10 Monte Carlo simulation: A stock price example

This example demonstrates how paths of a gBm (like those in the previous figure) can be produced via Monte Carlo simulation. Consider a stock, priced $S_0 = £1$ today, that pays no dividends. Using the past history of the stock's price, you estimate $\mu = 15\%$ and $\sigma = 30\%$ per annum. If the stock price follows a gBm

$$dS = \mu S dt + \sigma S dz \tag{9}$$

then, for any small discrete time step Δt, we can write

$$\begin{aligned} \Delta S &= \mu S \Delta t + \sigma S \varepsilon \left(\Delta t\right)^{\frac{1}{2}} \\ \varepsilon &\sim \text{i.i.d. } \mathcal{N}\left(0,1\right) \end{aligned}$$

Thus, we can simulate possible future stock price paths by:

1. Choose the magnitude of the time step, Δt. Suppose we choose $\Delta t = \frac{1}{365}$ (one day). Substitute values and write the above equations as

$$S_{t+1} - S_t = 0.15 S_t \frac{1}{365} + 0.3 S_t \varepsilon \left(\frac{1}{365}\right)^{\frac{1}{2}}$$

2. Draw a random number (ε) from a standard normal distribution.[‖] For a first draw $\varepsilon = 0.834$,

$$S_1 = £1 + 0.15 \times 1 \times \frac{1}{365} + 0.3 \times 1 \times 0.834 \times \left(\frac{1}{365}\right)^{\frac{1}{2}} = £1.013507$$

3. Drawing a new (independent) ε will produce S_2, and so on.[**]

[‖] This can be accomplished in MS Excel for example, through *Tools > Data Analysis > Random Number Generation*.

[**] HOMEWORK: If the same stock pays a continuous dividend yield δ, then $dS = (\mu - \delta) S dt + \sigma S dz$. Plot 10 different stock price paths for $\delta = 3\%$.

References

[1] L. Bachelier. *Theorie de la speculation*. PhD thesis, Annales de l'Ecole Normale Superieure, 1900.

[2] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[3] P. A. Samuelson. Rational theory of warrant pricing. *Industrial Management Review*, 6(Spring):13–31, 1965.