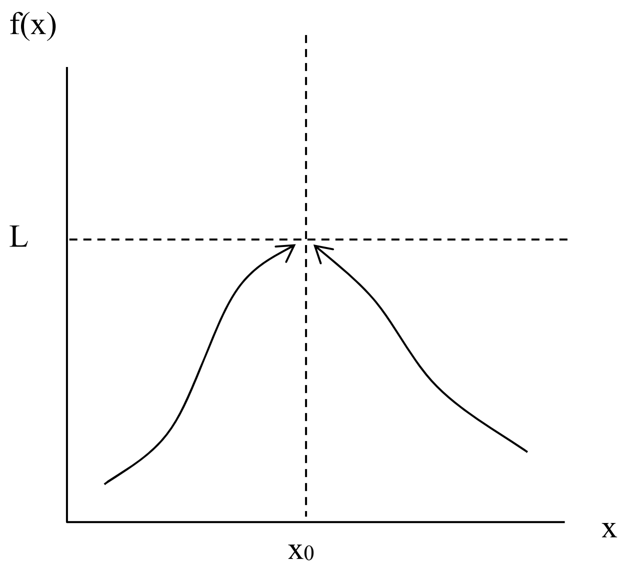
IV. Differentiation – Differential Calculus

1. Limits

We define as the **limit** of a function $y = f(x)$ as x approaches x_0 a number L in which the function converges. Informally the function has a limit L at an input x_0 if $f(x)$ is "close" to L whenever x is "close" to x_0. In other words, $f(x)$ becomes closer and closer to L as x moves closer and closer to x_0. In this case we write,

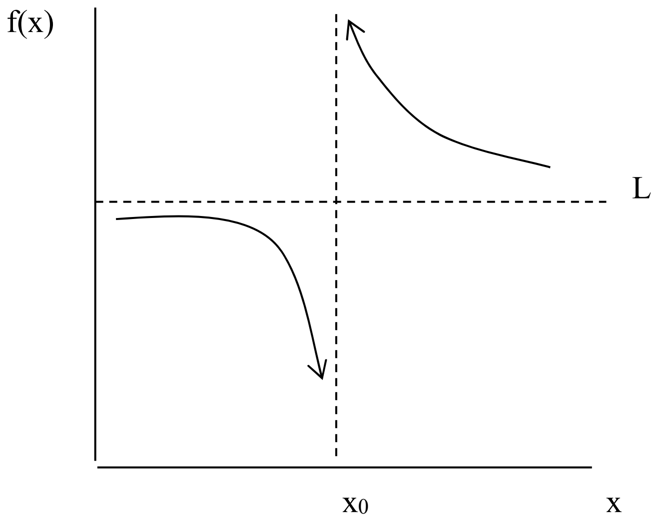
$$\lim_{x \to x_0} f(x) = L$$



In order for the limit to exist the function $f(x)$ should approach L both from the left and the right, that is,

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$$

In the following graph this is not true. As you observe the function converges to different numbers as x approaches x_0 from the left and the right. In this case the limit of f at x_0 does not exist.



Properties of limits:
Assume that $\lim_{x\to p} f(x) = a$, $\lim_{x\to p} g(x) = b$, then

- $\lim_{x\to p} \lambda f(x) = \lambda a$, for $\lambda \in \mathbb{R}$
- $\lim_{x\to p} (f(x) \pm g(x)) = a \pm b$
- $\lim_{x\to p} f(x)g(x) = ab$
- $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{a}{b}$, if $b \neq 0$
- $\lim_{x\to p} |f(x)| = |a|$
- $\lim_{x\to p} f(x)^n = a^n$

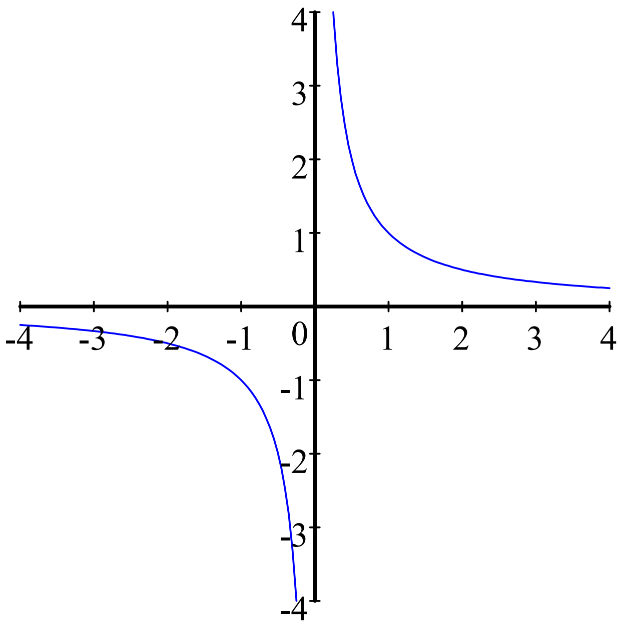
Limits can be used in order to determine the **asymptotes** of a function. The line y = a is the **horizontal asymptote** of a function f if:

$$\lim_{x\to +\infty} f(x) = a \text{ or } \lim_{x\to -\infty} f(x) = a$$

The line x = a is the **vertical asymptote** of a function f if:

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty$$

Example 13: Consider the function f(x) = 1/x.



Then we can prove that:

- $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, so 0 is a horizontal asymptote of the function.
- $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to 0^-} f(x) = -\infty$, so the function f is discontinuous at 0 and 0 is a vertical asymptote of the function.

2. Continuity

A function is **continuous** at x_0 if:

$$\lim_{x\to x_0} f(x) = f(x_0)$$

The above definition implies that for a function to be continuous at a specific point then the following conditions should be satisfied:

- The limit of f(x) at x_0 exists
- This limit is equal to $f(x_0)$

The function f is continuous at A if it continuous at every $x \in A$.

Using the properties of limits one can easily show that if the functions f and g are continuous at x_0 then the functions: $\lambda f, f \pm g, fg, \frac{f}{g}$ if $g(x_0) \neq 0, |f|$ and f^n are also continuous at x_0.

We can also define continuity using the left and right limits.
If f is defined on x_0 and $\lim_{x \to x_0^+} f(x) = f(x_0)$ the function is **right continuous** at x_0.
If f is defined on x_0 and $\lim_{x \to x_0^-} f(x) = f(x_0)$ the function is **left continuous** at x_0.
A function f is continuous at x_0 if it is both right and left continuous at this point, i.e,

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$$

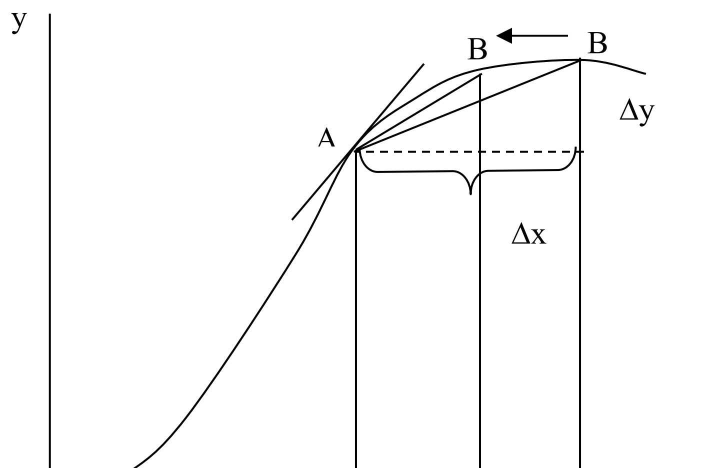
If a function is not continuous at a point x_0 it is called **discontinuous** at this point. From the above definition a function is discontinuous if one of the following conditions holds:

- $\lim_{x \to x_0^+} f(x), \lim_{x \to x_0^-} f(x)$ do not exist
- $\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$
- $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L \neq f(x_0)$

3. Definition of the first derivative

Assume a function y = f(x). Consider a point A with coordinates $(x_0,f(x_0))$ and B another point on the curve with coordinates (x,f(x)). The slope of the line AB is: $\frac{f(x)-f(x_0)}{x-x_0}$, that is the rate of change of y with respect to the change of x. As x approaches x_0 (or B approaches A) the line AB approaches the tangent line at the point A. The slope of this tangent line at A is the **first derivative** of the function f at x_0. It is also called the **instantaneous rate of change**. Formally this is defined as:

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$$



Alternatively, one could consider that the independent variable x changes by Δx, that is, from x_0 to $x_0 + \Delta x$. In this case the dependent variable would change from $f(x_0)$ to $f(x_0 + \Delta x)$. This change is equal to: $\Delta y = f(x_0 + \Delta x) - f(x_0)$. The mean change of y with respect to x in the interval $(x_0, x_0 + \Delta x)$ is:

$$\frac{f(x_0 + \Delta x) - f(x_0)}{x + \Delta x - x_0} = \frac{\Delta y}{\Delta x}$$

If the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

is finite then we say that the function f is **differentiable** at x_0. This limit is the first derivative of f at x_0. Setting $x = x_0 + \Delta x$ we obtain the previous definition of the first derivative.

If the function f is differentiable at a set A of its domain, then we define a new function on A that assigns to every point $x \in A$ the first derivative of f at x. This function is called **first derivative** or simply **derivative** of f and it is denoted as $f'(x)$ or $\frac{df(x)}{dx} \equiv \frac{dy}{dx}$.

Example 14: Consider the function $f(x) = 3x^2 - 4$. Then we have that:

$$\frac{\Delta y}{\Delta x} = \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x} =$$

$$= \frac{3(x_0^2 + 2x_0\Delta x + \Delta x^2) - 3x_0^2}{\Delta x} =$$

$$= \frac{6x_0\Delta x + 3\Delta x^2}{\Delta x} = 6x_0 + 3\Delta x$$

So, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 6x_0$.

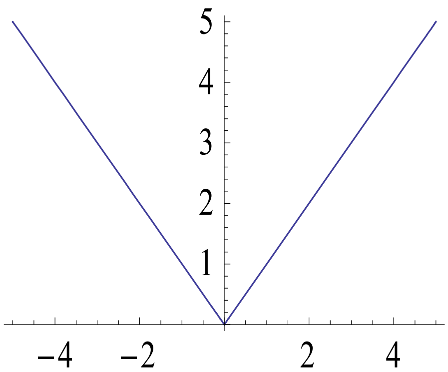
Since x_0 is an arbitrarily chosen point of the domain of f the last result implies that the derivative of f is:

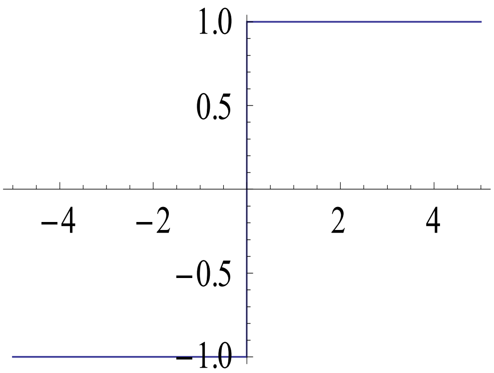
$$f'(x) = 6x$$

The above result implies that the instantaneous rate of change of y with respect to x is different at different points of the domain. In other words, how much would y change with respect to x depends on the value of x.

Theorem 1: If a function f is differentiable at a point x_0 of its domain, then f must also be continuous at x_0.

The opposite does not hold. Thus continuity of f is necessary but not sufficient for the differentiability of f at x_0. In other words, if f is discontinuous at x_0 then the derivative of f at x_0 does not exist. On the other hand, if f is continuous at x_0 the derivative may or may not exist.





4. Rules of differentiation

The following table gives the derivative of some well-known functions.

$f(x)$	$f'(x)$
c	0
x	1
cx^n	cnx^{n-1}
e^x	e^x
$\ln x$	$1/x$

The derivative has also the following properties:

- $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, if $g(x) \neq 0$
- $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(x)}$
- $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain rule)

The last property and the results of the previous table imply:

- $\frac{d}{dx}f(x)^n = nf(x)^{n-1}f'(x)$
- $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$
- $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

5. The instantaneous rate of growth

Assume that a variable y is a function of time t. For example the final amount of money which is compounded is a function of time, y = f(t).
The **instantaneous rate of growth** of y is defined as:

$$\frac{dy/dt}{y} = \frac{f'(t)}{f(t)}$$

which is also equivalent (see the last bullet) to:

$$\frac{d}{dt}\ln y$$

This last property enables us to determine the instantaneous rate of growth in many economic problems.

Example 15: Assume that $y(t) = y_0 e^{rt}$. Then we can write,

$$\ln y(t) = \ln y_0 + rt$$

and the instantaneous rate of growth equals:

$$\frac{d}{dt}\ln y = r$$

which is constant. If r > 0 the variable y exponentially increases with respect to t, while when r < 0 the variable exponentially decreases.

The notion of constant instantaneous growth rate is very familiar in finance. Assume that r is an interest rate. Then y(t) gives the amount of money one would collect from time 0 (when he/she invests y_0) to time t if this amount of money is continuously compounded. That is in every time period there is a flow of cash that increases your money by r. This r is equal to the percentage change of the variable in a very small period of time. Notice that the absolute change is not constant, actually it increases with respect to t, because the more money you invest the more you gain from them.

The exponential function that appears in the continuous compounding formula is related to the well-known discrete compounding formula

$$y(t) = y_0 \left(1 + \frac{r}{n}\right)^{nt}$$

This comes from the fact that:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

So, when the number of compounding during the year (given by n) approaches infinity the discrete compounding formula approaches the exponential function.

5. Monotonicity

Consider a function y = f(x). This function is called **increasing** (**decreasing**) over its domain if for every x_1, x_2 of the domain with $x_1 < x_2$ we have $f(x_1) \le f(x_2)$ ($f(x_1) \ge f(x_2)$). This is equivalent to:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \ge 0 (\le 0)$$

The function is called **strictly increasing** (**strictly decreasing**) if for $x_1 < x_2$ we have $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$).

Theorem 2: Consider a function f which is differentiable over its domain A. Then the following arguments hold:

- f is increasing on A *if and only if* $f'(x_0) \ge 0$ for every $x_0 \in A$.
- f is decreasing on A *if and only if* $f'(x_0) \le 0$ for every $x_0 \in A$.
- If $f'(x_0) > 0$ for every $x_0 \in A$ then f is strictly increasing on A.
- If $f'(x_0) < 0$ for every $x_0 \in A$ then f is strictly decreasing on A.

From the theorem it is obvious that if f is strictly increasing (decreasing) this does not imply that $f'(x_0) > 0$ ($f'(x_0) < 0$). For example the function $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.

6. Convexity

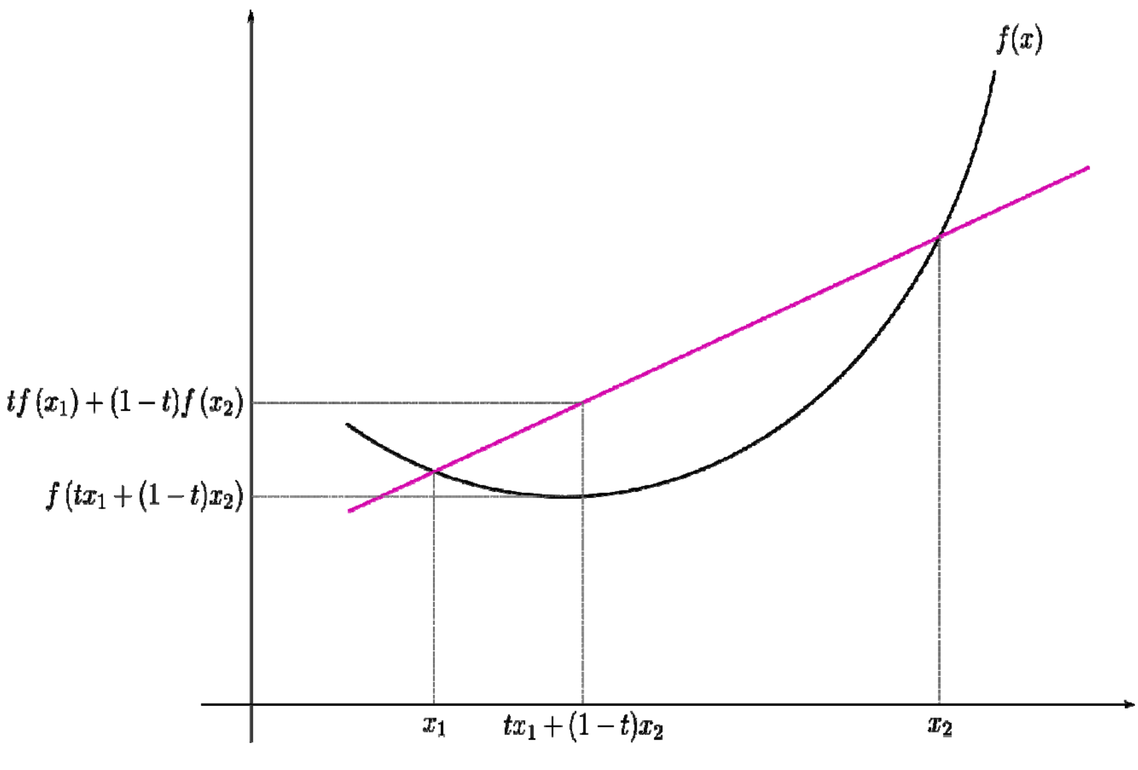
Consider the function y = f(x). This function is called **convex** if for any two points x_1 and x_2 in its domain and any $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

The function is called **concave** if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

If the order ≤ in the definition of convexity is replaced by the strict order < then one obtains a **strictly convex** function. Again, by inverting the order symbol, one finds a **strictly concave** function.



Theorem 3: Consider a function f twice differentiable on its domain A. Then the following arguments hold:

- f is convex on A *if and only if* $f''(x_0) \geq 0$ for every $x_0 \in A$.[3]
- f is concave on A *if and only if* $f''(x_0) \leq 0$ for every $x_0 \in A$.
- If $f''(x_0) > 0$ for every $x_0 \in A$ then f is strictly convex on A.
- If $f''(x_0) < 0$ for every $x_0 \in A$ then f is strictly concave on A.

7. Extrema

A large of number of problems in economics, finance and business administration are related to determine the maximum or the minimum point of a function. In some of these problems additional constraints should be imposed, leading to the constrained optimization of the function. We will examine these problems in the reminder of the course. Examples of such problems are:

- Find the optimal portfolio which maximizes return (or minimizes the risk)
- Determine asset prices and returns assuming a representative investor maximizing his/her utility function
- Determine the quantity of products that a firm should produce in order to maximize its profits
- Determine the optimal timing of investing or selling an asset

[3] The function $y = f''(x)$ is the second derivative of the function f. It can be defined as the first derivative of the first derivative function $y = f'(x)$.

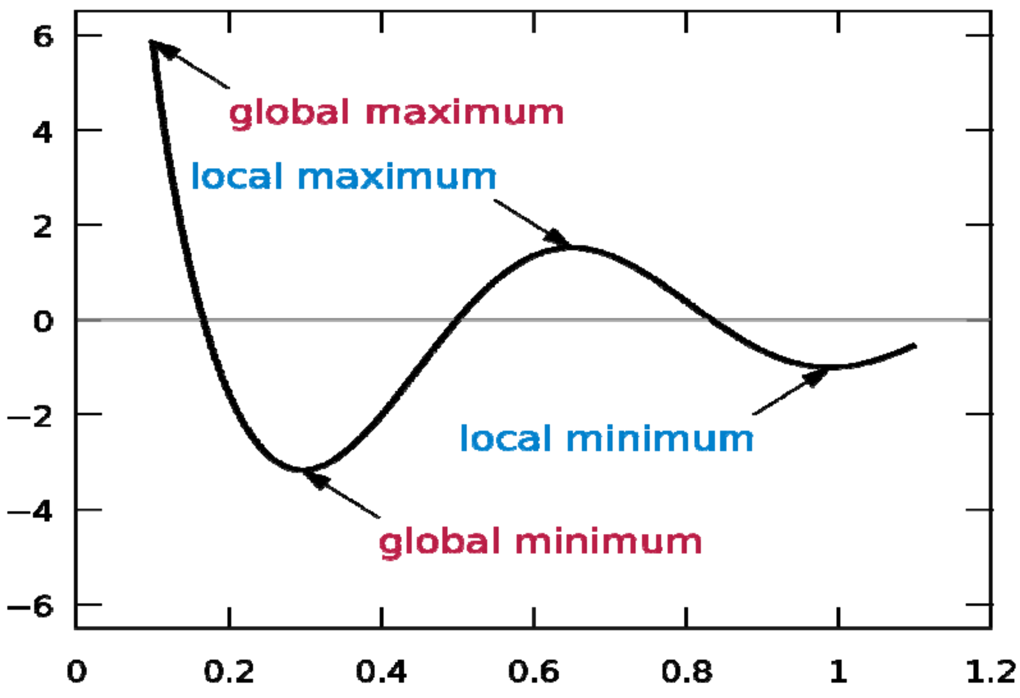
A function f is said to have a **local maximum** at x^*, if there exists some $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ when $|x - x^*| < \varepsilon$.

A function f is said to have a **local minimum** at x^*, if there exists some $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ when $|x - x^*| < \varepsilon$.

A function is said to have a **global maximum** at x^*, if $f(x^*) \geq f(x)$ for all x.

A function is said to have a **global minimum** at x^*, if $f(x^*) \leq f(x)$ for all x.

The local (global) minima and maxima are known as the **local (global) extrema**.



If the order $\leq$ or $\geq$ is replaced by the strict order $<$ or $>$ for all $x \neq x^*$ then the global and local maxima are unique.

Theorem 4: Consider a function f which is differentiable on the interior of its domain. If the point x^* is a local extremum of the function then $f'(x^*) = 0$.

Remarks:

- The above theorem implies that a point x^* with $f'(x^*) \neq 0$ cannot be a local extremum of f. To this end, the above condition is a *necessary condition for the existence of a local extremum*. It is also called **first order condition**.
- The opposite argument does not hold. For example the function f(x) = x^3 has $f'(0) = 0$ but 0 is not a local maximum or minimum. These points where the first derivative is equal to zero are called **stationary points**.
- The requirement that x^* belongs to the interior of the function domain is necessary. A function which is defined on a closed set could have local extrema on the bounds of this domain. For example the function f(x) = x^2 + 1 defined on the closed set [1,4] has minimum on x = 1 and maximum on x = 4, but $f'(1) = 2 \neq 0$ and $f'(4) = 8 \neq 0$.
- A function could have local extrema in the interior of its domain without being differentiable at this point. For example the function $f(x) = |x|$ is not differentiable on 0 but it has a local minimum on this point.

In general a function f defined over a closed and bounded domain A can have an extremum in the following points:

- Stationary points
- Bounds of the domain A

- Points where f is not differentiable
- Points where f is not continuous

The points where f can have an extremum are called **critical points** of f.

Consider that the function f has an extremum on the interior of its domain. Then the following theorem characterizes this extremum.

Theorem 5: Consider a function f twice differentiable on the interior of its domain and a point x^* of this interior with $f'(x^*) = 0$.

- If $f''(x^*) > 0$, then f has a local minimum on x^*.
- If $f''(x^*) < 0$, then f has a local maximum on x^*.

These conditions are known as the sufficient conditions for determining a local extremum. They are also called **second order conditions**.

The sufficient conditions of the above theorem does not cover the case where $f''(x^*) = 0$. For example the function $f(x) = x^4$ has $f'(0) = f''(0) = 0$ and 0 is a local minimum. On the other hand the function $f(x) = x^5$ has also $f'(0) = f''(0) = 0$ but 0 is just a point of inflexion. Thus we need a more general statement than theorem 5 to cover all the possible cases. This is given by the following theorem.

Theorem 6: Consider a function f n-differentiable on the interior of its domain and consider a point x^* of this interior for which:

$$f'(x^*) = f''(x^*) = ... = f^{(n-1)}(x^*) = 0, f^{(n)}(x^*) \neq 0$$

Then,

- If n is even then,
  - If $f^{(n)}(x^*) > 0$, then f has a local minimum on x^*
  - If $f^{(n)}(x^*) < 0$, then f has a local maximum on x^*
- If n is odd, then f has a point of inflexion on x^*.

For example the function $f(x) = x^6$ has:

$$f'(0) = f''(0) = ... = f^{(5)}(0) = 0$$

and $f^{(6)}(0) = 720 > 0$. Thus it has a local minimum on 0. On the other hand the function $f(x) = x^5$ has:

$$f'(0) = f''(0) = ... = f^{(4)}(0) = 0$$

and $f^{(5)}(0) = 120 > 0$. Thus it has a point of inflexion on 0.

The above theorems characterize local minima and maxima. Under certain conditions these local extrema are also the global extrema of a function. These are given in the following theorem.

Theorem 7: Consider a function f defined on a closed and bounded domain A.

- If f is continuous and concave (convex) on A, then every local maximum (minimum) is also a global maximum (minimum).
- If f is strictly concave (convex), then the global maximum (minimum) is unique.

Example 16: Consider the function $f(x) = \frac{1}{3}x^3 - 4x^2 + 2$ which is continuous and differentiable in the domain $\mathbb{R}$.

1. Find the local maxima and minima of f.

2. Find the intervals where f is increasing and decreasing.
3. Find the intervals where f is convex and concave.

Using the first order condition we specify the points which we may observe local extrema. Thus,

$$f'\left(x^*\right)=0 \Rightarrow \left(x^*\right)^2-8x^*=0 \Rightarrow x^*\left(x^*-8\right)=0$$

Thus, the stationary points of this function are $x^*=0$ and $x^*=8$. Using the second order condition we would specify if these points are local extrema or points of inflexion. We have that,

$$f''(x)=2x-8$$

Thus, $f''(0)=-8<0$ and $f''(8)=8>0$. So, in x = 0 the function has a local maximum and in x = 8 it has a local minimum.

To determine the intervals where f is increasing or increasing we should examine the sign of the first derivative. But we know that a function changes signs around its root. Solving for the first order condition we find that the roots of f' are 0 and 8. At the interval $\left(-\infty,0\right)$, $f'(x)>0$, so the function f is (strictly) increasing. At the interval $\left(0,8\right)$, $f'(x)<0$, and the function f is (strictly) decreasing. Finally, at the interval $\left(8,+\infty\right)$, $f'(x)>0$, so f is (strictly) increasing.

To determine the intervals where f is concave or convex we should examine the sign of the second derivative function. If we solve $f''(x)=0 \Rightarrow 2x-8=0 \Rightarrow x=4$, we find that f'' changes signs around 4. At the interval $\left(-\infty,4\right)$, $f''(x)<0$, so f is (strictly) concave, whereas at the interval $\left(4,+\infty\right)$, $f''(x)>0$, so f is (strictly) convex.

8. Profit maximization

The *cost function* C = C(Q) specifies the total cost C for the production of a quantity Q. We also define the *marginal cost* as

$$MC = \frac{dC}{dQ}$$

and the *average cost* as

$$AC = \frac{C}{Q}$$

According to economic theory the function C has the following characteristics:

- It is strictly increasing ($C'(Q) > 0$) because as the quantity increases the cost also increases.
- For small values of Q the cost increases with a decreasing rate, so $C''(Q) < 0$ for $Q < Q_1$, and then it increases with increasing rate, so $C''(Q) > 0$ for $Q > Q_1$. Thus Q_1 is a point of inflexion of C. This is due to the law of diminishing returns.

The *revenue function* R = R(Q) specifies the total income R earned by selling a quantity Q. We define the *marginal revenue* as

$$MR = \frac{dR}{dQ}$$

and the *average revenue* as

$$AR = \frac{R}{Q}$$

Consider a firm which wants to determine the quantity of products Q that maximizes its profits.
The *profit function* is defined as:

$$\pi(Q) = R(Q) - C(Q)$$

The first order condition determines that a local extremum Q^* exists when:

$$\pi'(Q^*) = 0 \Rightarrow R'(Q^*) = C'(Q^*) \Rightarrow MR = MC$$

that is, when marginal revenue is equal to marginal cost. The second order condition determines that this point correspond to a maximum if

$$\pi''(Q^*) < 0 \Rightarrow R''(Q^*) < C''(Q^*) \Rightarrow \frac{dMR}{dQ} < \frac{dMC}{dQ}$$

Thus at the point where the profit function is maximized the slope of the marginal revenue is smaller to the slope of the marginal cost.

Example 17: Assume that the firm is a *monopoly*. In this case the firm can determine itself the quantity and the price of the product. Even in this case the firm cannot impose an extremely high price because demand would decrease causing the decrease of profits. Assume that the relation between demand quantity Q and price P is determined by the function P = f(Q) for which $f'(Q) < 0$.

Assume that f(Q) = 250 – 20Q.

In this case the revenue function is equal to:

$$R(Q) = QP = Q(250 - 20Q) = 250Q - 20Q^2$$

and the marginal revenue equals:

$$MR = \frac{dR}{dQ} = 250 - 40Q$$

Assume that $C(Q) = \frac{1}{3}Q^3 - \frac{7}{2}Q^2 + 15Q + 100$

Then,

$$MC = \frac{dC}{dQ} = Q^2 - 7Q + 15$$

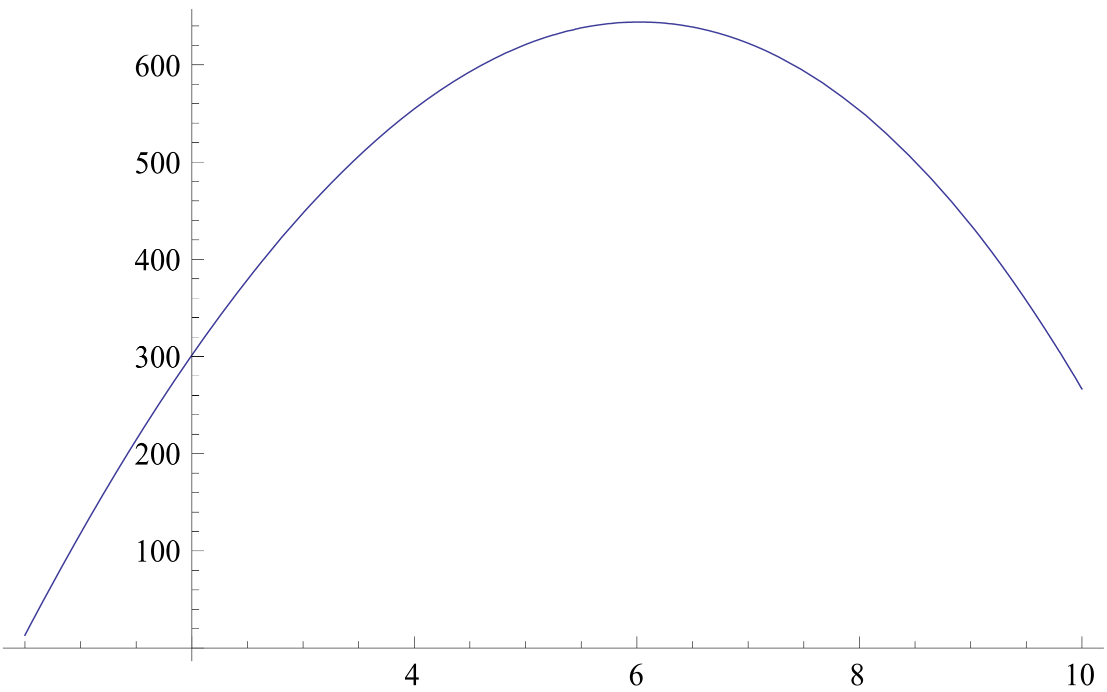
The first order condition gives:

$$MR = MC \Rightarrow 250 - 40Q = Q^2 - 7Q + 15 \Rightarrow Q^2 + 33Q - 235 = 0$$

The solutions of the last equation are Q_1 = -39.02 which is rejected and Q_2 = 6.02.
The second order condition is:

$$\frac{dMR}{dQ} < \frac{dMC}{dQ} \Rightarrow -40 < 2Q - 7$$

which holds for Q = 6.02, so at this point the profit of the firm is maximized. For this quantity demanded the price should be set at P = 129.6.



Example 18: Assume that the firm operates in *perfectly competitive market*. In this case it cannot influence the price of the product and the price is considered constant both for demanders and suppliers.
The revenue function is equal to: $R(Q) = \overline{P}Q$ and the marginal revenue is constant, $MR = \overline{P}$.
Assume now that the firm of the previous example operates in a perfectly competitive market with $\overline{P} = 50$.

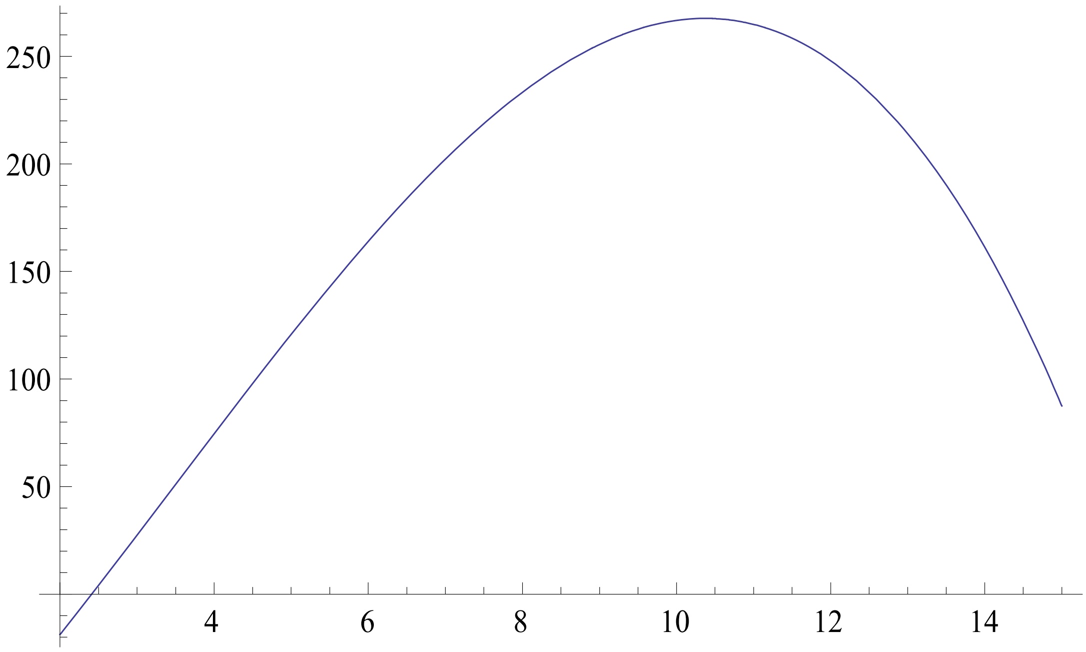
The first order condition is:

$$MR = MC \Rightarrow 50 = Q^2 - 7Q + 15 \Rightarrow Q^2 - 7Q - 35 = 0$$

The solutions of the last equation are Q_1 = -3.37 which is rejected and Q_2 = 10.37.

$$\frac{dMR}{dQ} < \frac{dMC}{dQ} \Rightarrow 0 < 2Q - 7$$

For Q_2 = 10.37 this condition is satisfied, so for this quantity produced the firm maximizes its profits.



9. Partial derivatives

In many real problems that we face in economics and finance a variable depends on a number of other variables. For example the demand for a specific car depends on its price, the prices of other cars in the same category, the income of consumers, the age of the car e.t.c. Thus we need to introduce **multivariate functions** in order to describe these types of relations. Their general form is:

$$y = f(x_1, x_2, ..., x_n) = f(x)$$

where y is the dependent variable and $x' = (x_1, x_2, ..., x_n)$ is the vector of independent variables.

A **partial derivative** measures the instantaneous rate of change of the dependent variable with respect to one independent variable assuming that all the others are constant. For the previous function f we can define n partial derivatives,

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}$$

The rules for calculating a partial derivative are the same to the rules of calculating a derivative and are given below:

- If $z = f(x)g(x)$, then $\frac{\partial z}{\partial x_j} = f(x)\frac{\partial g}{\partial x_j} + g(x)\frac{\partial g}{\partial x_j}$
- If $z = \frac{f(x)}{g(x)}$, $g(x) \neq 0$ then $\frac{\partial z}{\partial x_j} = \frac{f(x)\frac{\partial g}{\partial x_j} - g(x)\frac{\partial g}{\partial x_j}}{g(x)^2}$
- If $z = f(x)^n$ then $\frac{\partial z}{\partial x_j} = nf(x)^{n-1}\frac{\partial f}{\partial x_j}$
- If z = f(x,y) with x = g(u,v) and y = h(u,v) then, $\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$.

10. Constrained optimization

In several problems that we face in economics and finance we must maximize or minimize a multivariate function with respect to several constraints. For example,

- To maximize the production function with respect to budget constraints

- To maximize the expected utility of an investor with respect to budget constraints
- To minimize the risk of a portfolio with respect to the return expected by the investor.

The problem can be written as follows:

$$\begin{aligned}&\max_{x_1,\dots,x_n} f(x_1,\dots,x_n)\\&\text{s.t.c } g_j(x_1,\dots,x_n)=b_j,\ j=1,2,\dots,m.\end{aligned} \tag{3}$$

The following theorem helps us solve this problem.

Theorem 8: If $x^*=(x_1^*,x_2^*,\dots,x_n^*)'$ is a local extremum of the problem (3) then it exists a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_n^*)'$ such as

$$\frac{\partial f(x^*)}{\partial x_i}-\sum_{j=1}^{m}\lambda_j*\frac{\partial g_j(x^*)}{\partial x_i}=0,\ i=1,2,\dots,n$$

The above theorem implies that in order to find x^* and λ^* we must first write the **Lagrangian function**

$$L(x,\lambda)=f(x)+\sum_{j=1}^{m}\lambda_j(b_j-g_j(x))$$

where λ is the **Lagnange multiplier**. Then we must solve the following system of equations:

$$\frac{\partial L}{\partial x_i}=0,\ i=1,2,\dots,n$$

$$\frac{\partial L}{\partial \lambda_j}=0,\ j=1,2,\dots,m$$

The above system of equations determine the **sufficient first order conditions** for x^* to be a local extremum of (3).

We need to determine if this extremum is a maximum or a minimum. For ease of simplicity we will solve the problem for n = 2 and m = 1.
We define the **bordered Hessian matrix** as:

$$\overline{H}_L(x_1, x_2) = \begin{pmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 L}{\partial x_1^2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 L}{\partial x_2 \partial x_1} & \dfrac{\partial^2 L}{\partial x_2^2} \end{pmatrix}$$

If $\left|\overline{H}_L(x^*)\right| > 0$ then x* is a local maximum If $\left|\overline{H}_L(x^*)\right| < 0$ then x* is a local minimum. These are the **second order conditions**.[4]

Example 19: An individual's expected utility of end-of-period wealth W_1 can be written as

$$E\left[U\left(W_1\right)\right] = \sum \pi_s U\left(Q_s\right)$$

where Q_s = the number of pure securities paying a dollar if state s occurs. In this context, Q_s represents the number of state s pure securities the individual buys as well as his end-of-period wealth if state s occurs.
Now consider the problem we face when we must decide how to invest our initial wealth W_0 (how much of each pure security we should buy) in order to maximize the expected utility. We therefore face a problem of maximizing the expected utility subject to a wealth constraint. The problem can be written as:

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[4] In general the bordered Hessian matrix is of dimension $(n + m \times n + n)$. In this case x* is a local maximum if the matrix is negative definite. If the matrix is positive definite x* is a local minimum.

$$\max_{Q_s} \sum_s \pi_s U(Q_s)$$

subject to

$$\sum_s p_s Q_s = W_0$$

Our portfolio decision consists of choices we make for Q_s. To solve the above problem we construct the Lagrange function:

$$L = \sum_s \pi_s U(Q_s) - \lambda \left[\sum_s p_s Q_s - W_0 \right]$$

where λ is a Lagrange multiplier.
We take the first partial derivatives with respect to the unknown variables Q_s and the Lagrange multiplier and we set them equal to zero.

$$\frac{\partial L}{\partial Q_s} = \pi_s U'(Q_s) - \lambda p_s = 0 \text{ for every s}$$

$$\frac{\partial L}{\partial \lambda} = \sum_s p_s Q_s - W_0 = 0$$

Solving the above system of equations we obtain the solution to our problem.

Consider an investor with a logarithmic utility function of wealth, i.e., U(C) = lnC, and initial wealth \$5,000. Assume a two-state world where the pure security prices are \$0.4 and \$0.6 and the state probabilities are 1/3 and 2/3. The system of equations can be written as:

$$\frac{\partial L}{\partial Q_1} = \frac{1}{3Q_1} - 0.4\lambda = 0$$

$$\frac{\partial L}{\partial Q_2} = \frac{2}{3Q_2} - 0.6\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 5,000 - 0.4Q_1 - 0.6Q_2 = 0$$

Solving the first two equations we obtain:

$$Q_1 = \frac{1}{1.2\lambda}$$

$$Q_2 = \frac{1}{0.9\lambda}$$

Substituting these equations to the last equation we obtain:

$$\frac{0.4}{1.2\lambda} + \frac{0.6}{0.9\lambda} = 5,000 \Leftrightarrow \lambda = \frac{1}{5,000}$$

Therefore, the optimal investment choices are: Q_1 = 4,166.7 and Q_2 = 5,555.5.

In order to ensure that the above solutions define a local maximum we must calculate the bordered Hessian matrix. We have that,

$$\frac{\partial g}{\partial Q_1} = p_1, \frac{\partial g}{\partial Q_2} = p_2$$

and

$$\frac{\partial^2 L}{\partial Q_1^{\;2}} = -\frac{\pi_1}{Q_1^{\;2}}, \frac{\partial^2 L}{\partial Q_2^{\;2}} = -\frac{\pi_2}{Q_2^{\;2}}, \frac{\partial^2 L}{\partial Q_1 \partial Q_2} = 0$$

So,

$$\overline{H}_L = \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & -\pi_1 / Q_1^{\;2} & 0 \\ p_2 & 0 & -\pi_2 / Q_2^{\;2} \end{pmatrix}$$

and we can prove that $\left|\overline{H}_L\right| > 0$, so above solutions define a local maximum.

V. Integration

1. Indefinite integral

When we differentiate a function f we calculate the instantaneous rate of change at a point of its domain. When this rate of change is known and we want to find the function itself we take the opposite path and we must apply **integration.** Thus integration is the opposite of differentiation. If y = f(x) then $\frac{dy}{dx} = f'(x)$ and $\int f'(x)dx = f(x) + c$.

So in general we define the **indefinite integral** of a function f, denoted as $\int f(x)dx$, as a family of functions described by the formula

$$\int f(x)dx = F(x) + c, \text{ where } F'(x) = f(x)$$

Example 20: Consider the function f(x) = x^3. The first derivative function is $f'(x) = 3x^2$. So one could assume that the indefinite integral of f' is $\int f'(x)dx = x^3$. However, the first derivative function of f(x) = $x^3 + 2$ is also equal to $f'(x) = 3x^2$. So starting with f' we are not exactly sure about f. However, one thing is certain that the family of functions that have the same f' differs by a constant c.

In order to specify c a **boundary condition** should be given. If for example in the previous example we know that f(1) = 3 then,

$$f(x) = x^3 + c \Rightarrow f(1) = 1 + c \Rightarrow 3 = 1 + c \Rightarrow c = 2$$

So, $f(x) = x^3 + 2$.

2. Rules of integration

We now give the indefinite integrals of some basic functions:

- $\int a dx = ax + c$
- $\int ax^n dx = a\frac{1}{n+1}x^{n+1} + c, n \neq -1$
- $\int ax^{-1} dx = a\ln|x| + c$
- $\int a^{bx} dx = \frac{a^{bx}}{b\ln a} + c$
- $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

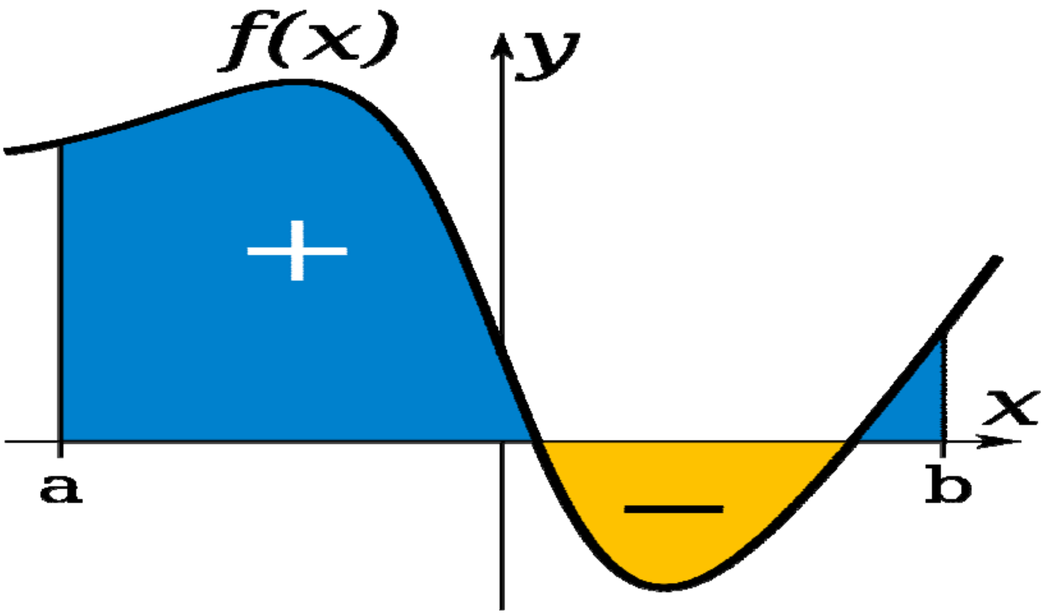
The following rules also hold:

- $\int af(x)dx = a\int f(x)dx$
- $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
- $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$ (integration by parts)
- $\int f(g(x))g'(x)dx = \int f(u)du$, with $u = g(x)$ (integration by substitution)

3. Definite integral

Indefinite integrals can be used to compute definite integrals. If $F'(x) = f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$



This number defines the net signed area of the region bounded by the graph of the function f, the x-axis and the vertical lines x = a and x = b.

The definite integrals have the following properties:

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$