Chapter 3: How to Calculate Present Values

3.1. Valuing long-lived assets

We have shown that the present value of an asset that produces a cash flow C_1 one year from now is:

$$PV_1 = \frac{C_1}{1+r_1},$$

where r_1 is the annual rate of return (OCC). Suppose that you will receive a certain cash inflow of \$100 next year and the interest rate is $r_1 = 7\%$. Then, the PV equals:

$$PV_1 = \frac{C_1}{1+r_1} = \frac{100}{1+0.07} = \$93.46$$

The present value of a cash flow C_2 two years from now is:

$$PV_2 = \frac{C_2}{(1+r_2)^2},$$

where r_2 is the *annual* rate of return for money invested in two years. Suppose that you get a cash inflow of \$200 in year 2, i.e. $C_2 = \$200$, and the interest rate of a two year deposit in the bank is $r_2 = 7.7\%$ (this means that a dollar invested in the bank deposit will grow to $1.077^2 = \$1.16$ by the end of two years). The PV of your 2-year cash flow equals:

$$PV_2 = \frac{C_2}{(1+r_2)^2} = \frac{200}{(1+0.077)^2} = \$172.42$$

In general, the present value of a cash flow C_t, t years from now is:

$$\boxed{PV_t = \frac{C_t}{(1+r_t)^t}}$$

where r_t is the annual rate of return for money invested in t years.

One of the most important properties of the present value rule is its *additivity*. This means that the present value of the cash flow A + B is the present value of the cash flow A plus the present value of the cash flow B. For example, suppose that you are offered an investment that produces a cash flow of \$100 in year 1 and a further cash flow \$200 in year 2. Like before, the one-year interest rate is $r_1 = 7\%$, while the two-year interest rate is $r_2 = 7.7\%$. The present value of the first year's cash flow is \$93.46, while the present value of the second year's cash flow is \$172.42. Thus, due to the additivity rule of present values, the total present value of the investment is:

$$PV = \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} = \$93.46 + \$172.42 = \$265.88$$

In general, if we assume a project that produces a cash flow C_1 the 1st year, C_2 the 2nd year, …., C_t the t-th year, …., then the present value of this project is:

$$\boxed{PV = \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} + \ldots + \frac{C_t}{(1+r_t)^t} + \ldots = \sum_{t=1}^{\infty} \frac{C_t}{(1+r_t)^t}}$$

This is called the **discounted cash flow** (or **DCF**) formula.
To find the net present value we add (the usually negative) initial cash flow, just as in Chapter 2:

$$NPV = C_0 + \sum_{t=1}^{\infty} \frac{C_t}{(1+r_t)^t}$$

Note that in the last formula every cash flow is discounted with his own discount rate r_t. This causes the problem that we should estimate a number of discount rates, especially for long-lived projects. Thus, for simplicity we will assume that $r_t = r$, that is, there is a unique discount rate for all periods.

Example: Assume a similar project like the one describer in Chapter 2, concerning office building. However, the construction will take two years to finish and the constructor is willing to accept a delayed payment. This means that the initial cash flow that we must pay is $170,000 ($50,000 for the land and $120,000 for the construction). The remaining $200,000 can be paid in years one and two (remember that the investment costs $370,000). The building will be worth $420,000 when completed. We can summarize the cash flows in the following table:

Period	0	1	2
Land	-50,000		
Construction	-120,000	-100,000	-100,000
Payoff			+420,000
Total cash flow	C_0 = - 170,000	C_1 = -100,000	C_2 = +320,000

If the interest rate is r = 5%, then the NPV is:

$$NPV = C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} =$$

$$= -170,000 - \frac{100,000}{1+0.05} + \frac{320,000}{(1+0.05)^2} = \$25,011$$

The NPV is positive; therefore we should accept this project.
In this example we have assumed that the cash flows are certain, and we use the interest rate for a bank deposit as the OCC. If they were risky, the OCC could be higher, say 12%. In this case the NPV would be negative.

3.2. Perpetuities and annuities

3.2.1. Perpetuities

Assets which offer a fixed income to perpetuity are called **perpetuities**. The present value of them is given by:

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

If C = $100,000 and r = 10% then PV = $1,000,000.

This approach is useful in at least two cases. One is the case of **consol bonds**. Unlike traditional bonds (we will examine them in Chapter 5) that repay principal at the end of a specific period (called maturity date), a consol never matures and pays a fixed coupon forever. Such bonds were issued by the British and the Canadian governments at the end of the 19^{th} century and at the beginning of the 20^{th}. The other case is **preferred stocks**. The owner of preferred stocks gets a fixed dollar payment, called a preferred dividend, every period; preferred stocks have also an infinite life.

Suppose that cash flows received in perpetuity are not constant but *increase* in a constant scale. This means that the cash flow received in year 1 is C = \$100,000, the cash flow received in year 2 is \$100,000(1+0.04) = \$104,000, the cash flow received in year 3 is \$104,000(1+0.04) = \$100,000$(1+0.04)^2$ = \$108,160 e.t.c. The factor g = 4% is the growing rate.

In general we can write that the cash flow received the year t as $C_t = C(1+g)^{t-1}$, where C is the cash flow received in year 1, and the present value of this **growing perpetuity** is:

$$PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + ... = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}$$

In the above example the PV = \$1,666,667.

Note that when the growth rate is equal or exceeds the discount rate, the present value of the growing perpetuity is infinite and thus cannot be computed.

We can also assume that the cash flows *decrease* in a constant scale. For example the first year cash flow could be C = \$100,000, then the second year cash flow is 100,000(1-0.04) = \$96,000, the third year cash flow is \$100,000$(1-0.04)^2$ = \$92,160 e.t.c. Like before the cash flow received the year t can be written as $C_t = C(1-g)^{t-1}$, and the present value is:

$$PV = \frac{C}{(1+r)} + \frac{C(1-g)}{(1+r)^2} + \frac{C(1-g)^2}{(1+r)^3} + ... = \sum_{t=1}^{\infty} \frac{C(1-g)^{t-1}}{(1+r)^t} = \frac{C}{r+g}$$

In the above example the PV = \$714,286.

3.2.2. Annuities

Assets that pay a fix amount C each year for a specified number of years denoted N (for example home mortgages, rental payment) are called **annuities**. The present value of them is given by:

$$PV = \sum_{t=1}^{N} \frac{C}{(1+r)^t} = C\left(\frac{1}{r} - \frac{1}{r(1+r)^N}\right)$$

The expression in parenthesis is called the *annuity factor*.

Suppose that in the previous example the project provides C = \$100,000 for N = 20 years. Then,

$$PV = C\left(\frac{1}{r} - \frac{1}{r(1+r)^N}\right) = 100,000\left(\frac{1}{0.1} - \frac{1}{0.1(1+0.1)^{20}}\right) = 100,000 \times 8.514 = \$851,400$$

Remember that the annuity formula assumes that the first payment occurs one period hence. If the first cash payment occurs immediately, we would need to discount each cash flow by one year less. Thus,

$$\begin{aligned} PV &= C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^{N-1}} \\ &= (1+r)\left(\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^N}\right) \\ &= (1+r)C\left(\frac{1}{r} - \frac{1}{r(1+r)^N}\right) \end{aligned}$$

Therefore, the present value of this investment, which is called **annuity due,** is equal to the present value of the annuity making the same number of annual payments multiplied by (1+r).
For example, if the project provides 20 annual payments starting immediately the present value would be: PV = 851,400(1+0.1) = \$936,540.

When the cash flows are not constant but increases (or decreases in a constant scale) then we have a **growing annuity**. Like before the cash flow received the year t is $C_t = C(1+g)^{t-1}$, where g is the growing factor. The present value is given by:

$$\boxed{PV = \sum_{t=1}^{N} \frac{C(1+g)^{t-1}}{(1+r)^t} = C\left(\frac{1}{r-g} - \frac{1}{r-g}\frac{(1+g)^N}{(1+r)^N}\right)}$$

For example, assume that C = \$100,000, r = 10% and g = 4%. Then,

$$PV = C\left(\frac{1}{r-g} - \frac{1}{r-g}\frac{(1+g)^N}{(1+r)^N}\right) = 100,000\left(\frac{1}{0.1-0.04} - \frac{1}{0.1-0.04}\frac{(1+0.04)^{20}}{(1+0.1)^{20}}\right) = \$1,123,839$$

Annuity formulas can also be used in the opposite way. We can use them to find the annual payment *given* the present value.

Example 1: Loan payment
Suppose that you take out a 4-year loan of \$1,000 from a bank, with an interest rate at 10% and annual payments. The yearly payments must be set so that they have a present value of \$1,000. Thus,

$$1,000 = C\left(\frac{1}{0.1} - \frac{1}{0.1(1+0.1)^4}\right) = C \times 3.1698 \Rightarrow C = \$315.47$$

This loan is an example of an **amortizing loan**. The amount C repaid each year to the bank is used to pay interest on the loan and part is used to reduce the amount of the loan. The next table illustrates the way this loan is paid.

Year	Year payment (1)	Year interest (2)	Amortization of loan (1) – (2)	End-of-year balance
0				1,000
1	315.47	100	215.47	784.53
2	315.47	78.45	237.02	547.51
3	315.47	54.75	260.72	286.79
4	315.47	28.68	286.79	0

Example 2: Choosing the rental payment

Assume that the office building is completed and it worth $420,000. Instead of sell it immediately you can rent it for 8 years and sell it after. What should be the minimum acceptable rental payment for the next 8 years?

We estimate that the prices of office buildings will increase by 3% per year and the opportunity cost of capital is 5%.

We have two alternative investment decisions. The first is to sell the office building immediately earning $420,000 and the second is to rent it for 8 years and sell it after. Therefore, the minimum acceptable rental payment should make the two investments equivalent, in other words the present value of both should be equal.

The present value of the first alternative is obviously $420,000.

The present value of the second is equal to the present value of renting the office building in years 1 through 8 plus the present value of the sale proceeds in year 8. At the end of year 8 the office building can be sold for $420{,}000(1+0.03)^8$ = $532,043. The present value of this sum is:

$$PV_2 = \frac{532,043}{(1+0.05)^8} = \$360,108$$

Thus, if we denote as PV_1 the present value of renting the building in years 1 through 8 then the following equation must hold:

$$420,000 = PV_1 + 360,108$$

or

$$PV_1 = 420,000 - 360,108 = \$59,892$$

Therefore, the present value of the eight annual rental payments that you receive must be at least equal to this amount. In other words,

$$C\left(\frac{1}{0.05} - \frac{1}{0.05(1+0.05)^8}\right) = \$59,892 \Rightarrow C \times 6.463 = \$59,892 \Rightarrow C = \$9,267$$

Thus, the minimum acceptable annual rental payment should be $9,267.

EXERCISES-3

1. The interest rate is 10%.

a. What is the PV of an asset that pays $1 a year in perpetuity?
b. What is the approximate PV of an asset that pays $1 a year in perpetuity beginning in year 8?
c. What is the approximate PV of an asset that pays $1 a year for each of the next seven years?

d. A piece of land produces an income that grows by 5% per annum. If the first year's income is $10,000, what is the value of the land?

2. A factory costs $800,000. You predict that it will produce an inflow of $170,000 a year for the next 10 years. If the OCC is 14%, what is the NPV of the factory? What will the factory worth at the end of 5 years?

3. Halcyon Lines is considering the purchase of a new bulk carrier for $8 million. The forecasted revenues are $5 million a year and operating costs are $4 million. A major refit costing $2 million will be required after both the fifth and the tenth years. After 15 years, the ship is expected to be sold for scrap at $1.5 million. If the discount rate is 8%, what is the ship's NPV?

4. You have just read an advertisement stating "Pay us $100 a year for 10 years and we will pay you $100 a year thereafter in perpetuity." If this is a fair deal, what is the rate of interest?

5. Yosuo Obuschi is 30 years old of age and his salary next year will be $40,000. He forecasts that his salary will increase at a steady rate of 5% per annum until his retirement at age 60.

a. If the discount rate is 8%, what is the PV of these future salary payments?
b. If Mr Obuschi saves 5% of his salary every year and invests these savings at an interest rate of 8%, how much will he have saved by age 60?
c. If he plans to spend these savings in even amounts over the subsequent 20 years, how much can he spend each year?

6. Yosuo Obuschi is now 60 years old and has a life expectancy of 12 more years. He wishes to invest $20,000 in an annuity that will make an annual payment until his death. If the interest rate is 8%, what income can he expect to receive each year?

7. The Zhangs family is saving to buy a boat at the end of 5 years. If the boat costs $20,000 and they can earn 10% a year on their savings, how much do they need to put aside at the end of years 1 through 5?

8. Auto company A is offering free credit on a new $10,000 car. You pay $1,000 down and then $300 a month for the next 30 months. Auto company B does not offer free credit but will give you $1,000 off the list price. If the interest rate is 10% a year, which company is offering the better deal?

9. Suppose that you take out a $200,000, 20-year mortgage loan to buy a house. The interest rate on the loan is 6%, and payments on the loan are made annually at the end of each year.

a. What is your annual payment on the loan?
b. Construct a mortgage amortization table in Excel, showing the interest payment, the amortization of the loan and the loan balance for each year.
c. What fraction of your initial loan payment is interest? What about the last payment? What fraction of the loan has been paid off after 10 years? Why is the fraction less than half?

10. You are trying to assess the value of a store that is up for sale. You expect that the store will generate a cash flow of $100,000 next year and is expected to have growth of 5% a year in perpetuity.

a. If the discount rate is 10%, what would your assessment be of the value of the store?
b. What would the growth rate need to be to justify a price of $2.5 million for this store?