6.3

Monday, April 29, 2024 4:40 PM

$$E(X) \triangleq \sum_{x \in S_X} x\, p_X(x)$$

ΚΕΦΑΛΑΙΟ 4

$Y = g(X)$ $(Y = X^2)$

$$E(Y) \triangleq \sum_{y \in S_Y} y\, p_Y(y)$$

ΘΕΩΡΗΜΑ

$$= \sum_{x \in S_X} g(x) \cdot p_X(x)$$

$$\left(E(X^2) = \sum_{x \in S_X} x^2 p_X(x) \right)$$

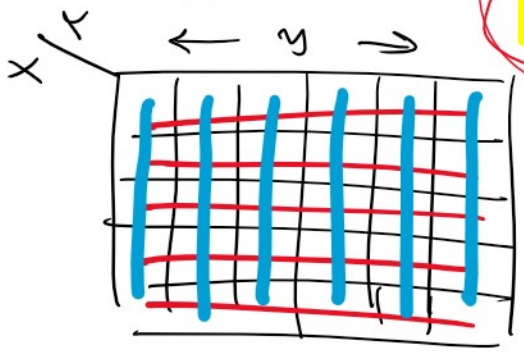
ΚΕΦΑΛΑΙΟ 6

X, Y S_X, S_Y $Z = g(X,Y)$

$Z = X+Y$
$Z = |X-Y|$
$Z = X \cdot Y$

$$E(Z) = E(g(X,Y)) =$$

$$\left(E(Z) \triangleq \sum_{z \in S_Z} z \cdot p_Z(z) \right)$$



$$\sum_{x \in S_X,\, y \in S_Y} g(x,y)\, p_{XY}(x,y)$$

$$= \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y)\, p_{XY}(x,y)$$

$$= \sum_{y \in S_Y} \sum_{x \in S_X} g(x,y)\, p_{XY}(x,y)$$

y \ x	0	1	2	3	$p_Y(y)$
0	1/6	1/12	1/12	1/12	15/36
1	1/12	1/18	1/36	1/36	7/36
2	1/12	1/36	1/18	1/36	7/36
3	1/12	1/36	1/36	1/18	7/36
$p_X(x)$	15/36	7/36	7/36	7/36	

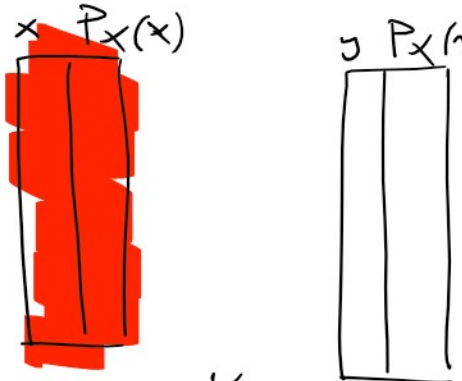
$Z = |X-Y|$

$$E(Z) = \frac{1}{6}|0-0| + \frac{1}{12}|1-0| + \frac{1}{12}|2-0| + \frac{1}{12}|3-0|$$
$$+ \frac{1}{12}|1-0| + \frac{1}{18}|1-1| + \frac{1}{36}|2-1| + \frac{1}{36}|3-1|$$
$$+ \frac{1}{12}|2-0| + \frac{1}{36}|2-1| + \frac{1}{18}|2-2| + \frac{1}{36}|3-2|$$
$$+ \frac{1}{12}|3-0| + \frac{1}{36}|3-1| + \frac{1}{36}|3-2| + \frac{1}{18}|3-3|$$

$$= \left(\frac{1}{12}+\frac{1}{12}+\frac{4}{36}\right)\cdot 1 + 2\cdot\left[\frac{2}{12}+\frac{2}{36}\right] + 3\cdot\left[\frac{2}{12}\right]$$

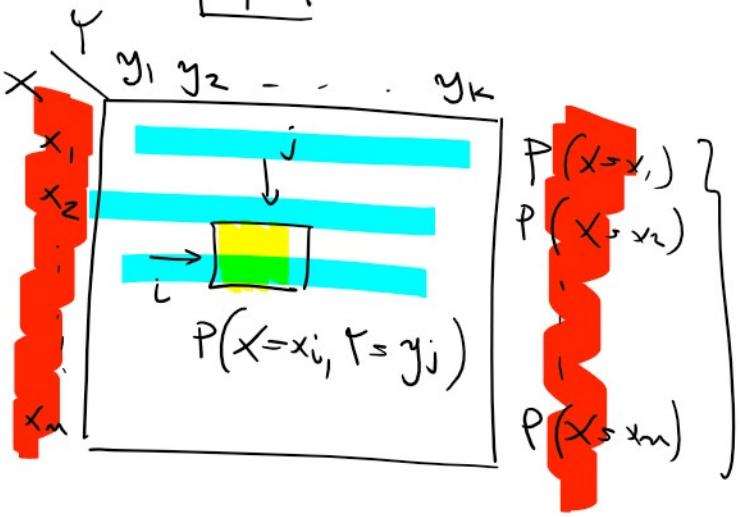
ΕΠΑΝΑΛΗΨΗ

$$\frac{10}{36} + \frac{16}{36} + \frac{18}{36} = \frac{44}{36} = \frac{11}{9}$$



$x \in S_X, y \in S_Y$

$P_{XY}(x,y)$



ΑΣΚΗΣΗ 43

ΑΣΚΗΣΗ 43

5 ΠΕΤΟΝΙΑ → 2 ΚΑΦΕ, 3 ΧΑΡΑΚΤΗΡΑ

CDN

X = # ΚΟΥΜΑΤΑ ΓΙΑ ΤΟ 1ο

Y = # ΚΟΥΜΑΤΑ ΓΙΑ ΤΟ 2ο

X \ Y	1	2	3	4	
1	1/10	1/10	1/10	1/10	4/10
2	1/10	1/10	1/10	0	3/10
3	1/10	1/10	0	0	2/10
4	1/10	0	0	0	1/10
	4/10	3/10	2/10	1/10	

$S_X = \{1, 2, 3, 4\}$

$S_Y = \{1, 2, 3, 4\}$

XXXKK	X=4, Y=1
XXKXK	X=3, Y=2
→ XKXXK	X=2, Y=3
KXXXK	X=1, Y=4
KXXKX	X=1, Y=3
KXKXX	X=1, Y=2
KKXXX	X=1, Y=1
XXKKX	X=3, Y=1
XKKXX	X=2, Y=1
XKXKX	X=2, Y=2

$$P(X=1, Y=1) = P(X=1)\, P(Y=1 \mid X=1) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$$

$$P(X=1, Y=1) = \frac{1}{10}$$

$$P(X=2, Y=2) = P(X_1 K_2 X_3 K_4)$$

$$= P(X_1)\, P(K_2 \mid X_1)\, P(X_3 \mid X_1 K_2)\, P(K_4 \mid X_1 K_2 X_3)$$

$$= \frac{3}{5} \quad \frac{2}{4} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{2} = \frac{1}{10}$$

ΛΗΜΜΑ 6.2

X, Y $P_{XY}(x,y)$ ΝΟΡΜΑ

1) $Z = g(X,Y)$ $(|X-Y| \quad X \cdot Y \quad X+Y)$

$E(Z) \triangleq \sum z\, P_Z(z)$

$$E(Z) \triangleq \sum_{z \in S_Z} z\, P_Z(z)$$

ΘΕΩΡΗΜΑ

$$= \sum_{\substack{x \in S_X \\ y \in S_Y}} g(x,y)\, P_{XY}(x,y)$$

$Z = aX + bY + c$

2) $E(aX+bY+c) = aE(X) + bE(Y) + c$

ΑΠΟΔΕΙΞΗ

$$E(aX+bY+c) = \sum_{\substack{x \in S_X \\ y \in S_Y}} (ax+by+c)\, P_{XY}(x,y)$$

$$= \sum_{\substack{x \in S_X \\ y \in S_Y}} ax\, P_{XY}(x,y) + \sum_{\substack{x \in S_X \\ y \in S_Y}} by\, P_{XY}(x,y) + \sum_{\substack{x \in S_X \\ y \in S_Y}} c\, P_{XY}(x,y)$$

$aE(X)$ $\qquad$ $bE(Y)$

$$a \sum_{x \in S_X} \sum_{y \in S_Y} x\, P_{XY}(x,y) = (*)$$

$$c \sum_{\substack{x \in S_X \\ y \in S_Y}} P_{XY}(x,y) = c$$

$$\sum_{\substack{x \in A \\ y \in B}} f(x,y) = \sum_{x \in A} \sum_{y \in B} f(x,y)$$

$= P_X(x)$

$$(*) \quad a \sum_{x \in S_X} \left(x \sum_{y \in S_Y} P_{XY}(x,y) \right)$$

$$= a \sum_{x \in S_X} x\, P_X(x) = aE(X)$$

$x \in S_X$

3) $E\left(\sum_{i=1}^{N} a_i g_i(X,Y) \right) =$

$$\sum_{i=1}^{N} a_i \, E\left(g_i(X,Y) \right)$$

ΠΑΡΑΔΕΙΓΜΑ 6.10

x / y	0	1	2	3	4	5	6	7	8	$p_Y(y)$
0	1/18	1/18	1/18	1/27	1/27	1/27	0	0	0	5/18
1	1/18	1/18	1/18	1/27	1/27	1/27	1/18	1/18	1/18	8/18
2	0	0	0	1/27	1/27	1/27	1/18	1/18	1/18	5/18
$p_X(x)$	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	

X = ΒΑΘΜΟΣ Τ.Ε. $E(X) = 4$

Y = ΒΑΘΜΟΣ ΠΡΟΦΟΡΑΣ $E(Y) = 1$

$$E(X) = 0 \cdot \frac{1}{9} + 1 \cdot \frac{1}{9} + 2 \cdot \frac{1}{9} + \dots + 8 \cdot \frac{1}{9} =$$

$$\frac{(1+2+\dots+8)}{9} = \frac{8 \cdot 9}{2 \cdot 9} = 4$$

$$E(Y) = 0 \cdot \frac{5}{18} + 1 \cdot \frac{8}{18} + 2 \cdot \frac{5}{18} = 1$$

$$E(X+Y) = E(X) + E(Y) = 4 + 1 = 5$$

ΑΣΚΗΣΗ 31

(γ) T_1 : ΤΟ ΒΡΥΧΙΟ

T_{100} : ΧΤΑΠΟ ΔΙ

Y_1 : ΦΟΡΕΣ ΣΤΟΝ T_1 $E(Y_1) = 1$

Y_2 : ΦΟΡΕΣ ΤΟΝ T_2 $E(Y_2) = A$

⋮

Y_{100} : ΦΟΡΕΣ ΣΤΟΝ T_{100} $E(Y_{100}) = A$

$$Y_1 + Y_2 + \dots + Y_{100} = Z \sim \text{BERN}\left(\frac{1}{100}\right)$$

$$E(Y_1 + Y_2 + \dots + Y_{100}) = E(Z)$$

$$E(Y_1) + E(Y_2) + \dots + E(Y_{100}) = E(Z)$$

$$A + \dots + A = 100$$

$$1 + 99A = 100 \Rightarrow A = 1$$

ΠΑΡΑΔΕΙΓΜΑ 6.11

X, Y = ΣΗΜΑΙΑ ΑΠΟ ΑΡΤΙΑ ΖΑΡΙΑ

$W = \max\{X, Y\}$

$Z = \min\{X, Y\}$

$$P = 2W + 3Z - 15$$

$$E(P) = E(2W + 3Z - 15) =$$

$$= 2E(W) + 3E(Z) - 15$$

$$= \frac{55}{36}$$

ΟΡΙΣΜΟΣ 6.2

X, Y

$E(X) = \mu_X$ VAR(X), σ_X

$E(Y) = \mu_Y$ VAR(Y), σ_Y

ΣΥΝΔΙΑΚΥΜΑΝΣΗ (COVARIANCE)

$$\text{COV}(X, Y) \stackrel{\Delta}{=} E[g(X, Y)] = E[(X - \mu_X)(Y - \mu_Y)]$$

ΠΑΡΑΔΕΙΓΜΑ 6.12

Y

X \ Y	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

$\mu_X = \frac{2}{3}$

$\mu_Y = \frac{2}{3}$

$Cov(X,Y) =$

$$\frac{1}{9}\left(0-\frac{2}{3}\right)\left(0-\frac{2}{3}\right)+\frac{2}{9}\left(1-\frac{2}{3}\right)\left(0-\frac{2}{3}\right)+\frac{1}{9}\left(2-\frac{2}{3}\right)\left(0-\frac{2}{3}\right)$$

$$+\frac{2}{9}\left(1-\frac{2}{3}\right)\left(0-\frac{2}{3}\right)+\frac{2}{9}\left(1-\frac{2}{3}\right)\left(1-\frac{2}{3}\right)+0\cdots$$

$$+\frac{2}{9}\left(2-\frac{2}{3}\right)\left(0-\frac{2}{3}\right)+0\cdots \quad ; \quad 0\cdots \quad = -\frac{2}{9}$$

LEMMA 6.3

1) $Cov(X,X) = VAR(X)$ $\quad Y = X$

$$Cov(X,X) = E\left[(X-\mu_X)(X-\mu_X)\right] =$$

$$E\left[(X-\mu_X)^2\right] = VAR(X)$$

2) $Cov(X,-X) = -VAR(X)$ $\quad Y = -X$, $E(Y) = -E(X)$

$$Cov(X,-X) = E\left[(X-\mu_X)(-X-(-\mu_X))\right]$$

$$= E\left[-(X-\mu_X)(X-\mu_X)\right] =$$

$$-E\left[(X-\mu_X)^2\right] = -VAR(X)$$

3) $Cov(X,Y) \triangleq E\left[(X-\mu_X)(Y-\mu_Y)\right]$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$Cov(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$\text{cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E\left[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y\right]$$

$$= E[XY] - E[\mu_X Y] - E[\mu_Y X] + E[\mu_X \mu_Y]$$

$$= E[XY] - \mu_X E[Y] - \mu_Y E(X) + \mu_X \mu_Y$$

$$= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E[XY] - E(X)E(Y)$$

4) $$\text{VAR}(X+Y) = \text{VAR}(X) + \text{VAR}(Y) + 2\,\text{cov}(X,Y)$$

$$\left(E(X+Y) = E(X) + E(Y)\right)$$

$$\text{VAR}(X+Y) = E\left[(X+Y)^2\right] - \left(E(X+Y)\right)^2$$

$$= E\left[X^2 + Y^2 + 2XY\right] - \left(E(X) + E(Y)\right)^2 =$$

$$E(X^2) + E(Y^2) + 2E(XY) - \left(E(X)\right)^2 - \left(E(Y)\right)^2 - 2E(X)E(Y)$$

$$\text{VAR}(X) + \text{VAR}(Y) + 2\,\text{cov}(X,Y)$$

X, Y

$\mu_X \quad \sigma_X \quad \mu_Y, \sigma_Y$

$$\text{cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_X)\right]$$

$$= E[XY] - E[X]E[Y]$$

ΠΑΡΑΔΕΙΓΜΑ 6.13

Y \ X	1	2	3	4	5	6	$p_Y(y)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6

3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6	

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0$$

$$\frac{49}{4} \quad - \quad \frac{7}{2} \quad \frac{7}{2}$$

$$E(XY) = \frac{1}{36}\cdot 1\cdot 1 + \frac{1}{36}\cdot 1\cdot 2 + \dots + \frac{1}{36}\cdot 1\cdot 6$$

$$+ \frac{1}{36}\cdot 2\cdot 1 + \dots$$

$$= \frac{1}{36}(1+2+3+\dots+6)\times(1+2+3+\dots+6)$$

$$= \frac{49}{4}$$

x / y	1	2	3	4	5	6	$p_Y(y)$
1	1/6	0	0	0	0	0	1/6
2	0	1/6	0	0	0	0	1/6
3	0	0	1/6	0	0	0	1/6
4	0	0	0	1/6	0	0	1/6
5	0	0	0	0	1/6	0	1/6
6	0	0	0	0	0	1/6	1/6
$p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6	

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{35}{12}$$

$$\frac{91}{6} \quad \frac{7}{2} \quad \frac{7}{2}$$

$$E(XY) = \frac{1}{6}\cdot 1\cdot 1 + \frac{1}{6}\cdot 2\cdot 2 + \frac{1}{6}\cdot 3\cdot 3 + \dots + \frac{1}{6}\cdot 6\cdot 6 = \frac{91}{6}$$

$W = \max(X,Y)$

$Z = \min(X,Y)$

w / z	1	2	3	4	5	6	$p_Z(z)$
1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
2	0	1/36	2/36	2/36	2/36	2/36	9/36
3	0	0	1/36	2/36	2/36	2/36	7/36
4	0	0	0	1/36	2/36	2/36	5/36
5	0	0	0	0	1/36	2/36	3/36
6	0	0	0	0	0	1/36	1/36
$p_W(w)$	1/36	3/36	5/36	7/36	9/36	11/36	

$Cov(W,Z) = E[WZ] - E[W]E[Z] = \frac{1225}{1296}$

$p_W(w)$	1/36	3/36	5/36	7/36	9/36	11/36

$$Cov(W,Z) = E[WZ] - E[W]E[Z] = \frac{1225}{1291}$$

$$\frac{4}{49} \quad \frac{161}{36} \quad \frac{91}{36}$$

$$E[WZ] = \frac{1}{36}\left[1\cdot 1 + 2\cdot 2 + \dots + 6\cdot 6\right] + \frac{2}{36}\left[1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 1\cdot 5 + 1\cdot 6 + 2\cdot 3 + 2\cdot 4 + \dots\right]$$

$$= \frac{49}{4}$$

ΠΑΡΑΔΕΙΓΜΑ 6.14

x / y	0	1	2	3	4	5	6	7	8	$p_Y(y)$
0	1/18	1/18	1/18	1/27	1/27	1/27	0	0	0	5/18
1	1/18	1/18	1/18	1/27	1/27	1/27	1/18	1/18	1/18	8/18
2	0	0	0	1/27	1/27	1/27	1/18	1/18	1/18	5/18
$p_X(x)$	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	

$$Cov(X,Y) = E[XY] - E[X]\cdot E[Y] = 1$$

$$5 \quad 4 \quad 1$$

$$E[XY] = \frac{1}{18}\cdot 1\cdot 1 + \frac{1}{18}\cdot 1\cdot 2 + \frac{1}{27}\cdot 1\cdot 3 + \dots = 5$$

ΠΑΡΑΔΕΙΓΜΑ 6.16

ΣΥΝΤΕΛΕΣΤΗΣ ΣΥΣΧΕΤΙΣΗΣ

ΟΡΙΣΜΟΣ → $\rho_{X,Y} = \dfrac{Cov(X,Y)}{[VAR(X)\,VAR(Y)]^{\frac{1}{2}}} = \dfrac{Cov(X,Y)}{\sigma_X\,\sigma_Y}$

ΙΔΙΟΤΗΤΑ → $|\rho_{XY}| \le 1$

Ν.Δ.Ο.

$Y = aX + b \Rightarrow \rho_{XY} = \pm 1$

$(a \neq 0)$

$Cov(X,Y) = E(XY) - E(X)E(Y) =$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) =$$

$$E(X(aX+b)) - E(X)E(aX+b) =$$

$$aE(X^2) + bE(X) - a(E(X))^2 - bE(X)$$

$$= a\left[E(X^2) - (E(X))^2\right] = a\,\mathrm{VAR}(X)$$

$$\mathrm{VAR}(Y) = \mathrm{VAR}(aX+b) = a^2\,\mathrm{VAR}(X)$$

$$\mathrm{VAR}(aX+b) = E\left[(aX+b)^2\right] - \left(E[aX+b]\right)^2$$

$$= E\left[a^2X^2 + b^2 + 2abX\right] - \left(aE(X) + b\right)^2$$

$$= a^2E(X^2) + b^2 + 2abE(X) - a^2(E(X))^2 - b^2 - 2abE(X)$$

$$= a^2\left[E(X^2) - (E(X))^2\right] = a^2\,\mathrm{VAR}(X)$$

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{VAR}(X)\,\mathrm{VAR}(Y)}} = \frac{a\,\mathrm{VAR}(X)}{\sqrt{\mathrm{VAR}(X)\,a^2\,\mathrm{VAR}(X)}}$$

$$= \frac{a}{\sqrt{a^2}} = \frac{a}{|a|} = \begin{cases} 1, & a>0, \\ -1, & a<0. \end{cases}$$