CHAPTER 5 - DUALITY

$$\left\{\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0, & i=1,\dots,m \\ & h_i(x) = 0, & i=1,\dots,p \end{array}\right\} \quad (5.1)$$

STANDARD FORM OPTIMIZATION PROBLEM. DOES NOT NEED TO BE CONVEX.

$x \in \mathbb{R}^n$. $\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$

DEFINITION: LAGRANGIAN $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

$\operatorname{dom} L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$. λ_i: LAGRANGE MULTIPLIER ASSOCIATED WITH i-TH INEQUALITY $f_i(x) \le 0$. ν_i: LAGRANGE MULTIPLIER ASSOCIATED WITH i-TH EQUALITY CONSTRAINT $h_i(x) = 0$

λ, ν: DUAL VARIABLES OR LAGRANGE MULTIPLIER VECTORS

DEFINITION: (LAGRANGE) DUAL FUNCTION $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$.

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left[f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right]$$

$\operatorname{dom} g = \{(\lambda, \nu) : g(\lambda, \nu) > -\infty\}$

IT IS ALWAYS CONCAVE, AS THE INFIMUM OF LINEAR FUNCTIONS, IRRESPECTIVE OF f_0, f_i, h_i. IT COULD BE $-\infty$.

5.1.3

BASIC PROPERTY OF $g(\lambda, \nu)$:

$$\forall \lambda \ge 0, \forall \nu, \quad g(\lambda, \nu) \le p^*$$

PROOF: LET $\tilde{x}$ FEASIBLE. THEN

$$L(\tilde{x},\lambda,\nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \underbrace{\lambda_i}_{\succeq 0} \underbrace{f_i(\tilde{x})}_{\le 0} + \sum_{i=1}^{p} \nu_i \underbrace{h_i(\tilde{x})}_{= 0} \le f_0(\tilde{x})$$

$\Rightarrow$

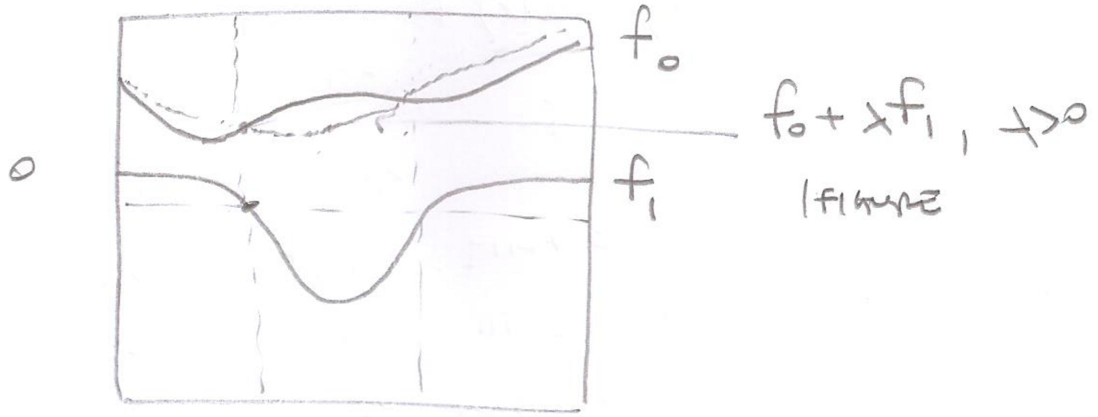
$$g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu) \le L(\tilde{x},\lambda,\nu) \le f_0(\tilde{x})$$

THIS HOLD for ALL $\tilde{x}$, SO $g(\lambda,\nu) \le p^*$

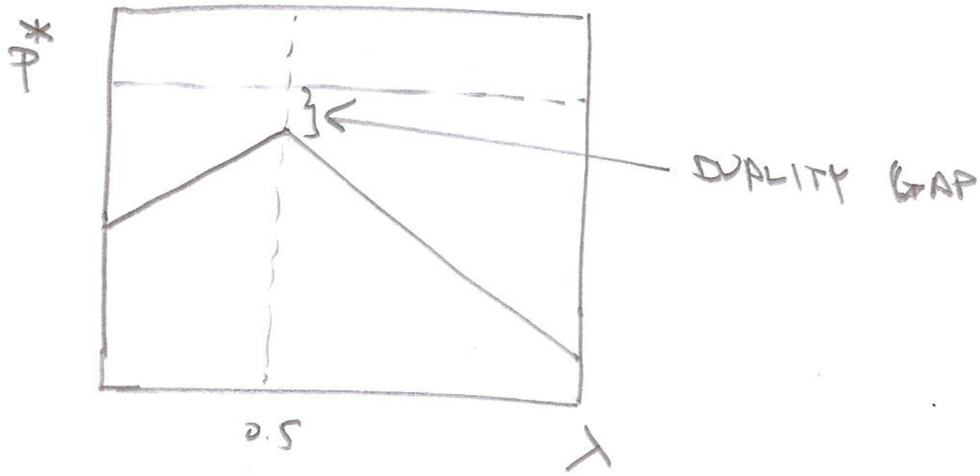
DEFINITION

ANY PAIR (λ,ν) WITH $\lambda \succeq 0$, $(\lambda,\nu) \in \mathrm{dom}\, g$ IS DUAL FEASIBLE

EXAMPLE:



(FIGURE



5.1.4 LINEAR APPROXIMATION INTERPRETATION

LET $I_-(u) = \begin{cases} 0, & u \le 0 \\ \infty, & u > 0 \end{cases}$ $\quad I_0(u) = \begin{cases} 0, & u = 0 \\ \infty, & u \ne 0 \end{cases}$

$\mathbb{R} \to \mathbb{R}$

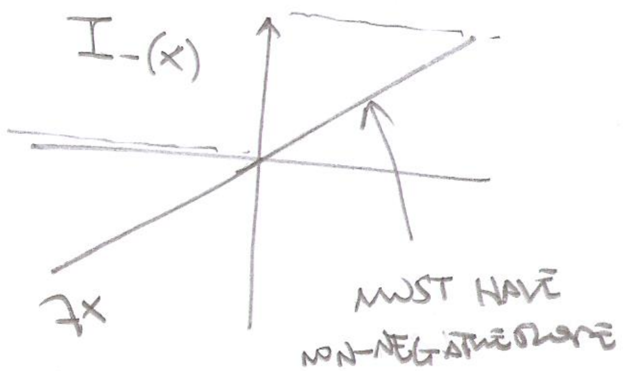
THEN ORIGINAL PROBLEM:

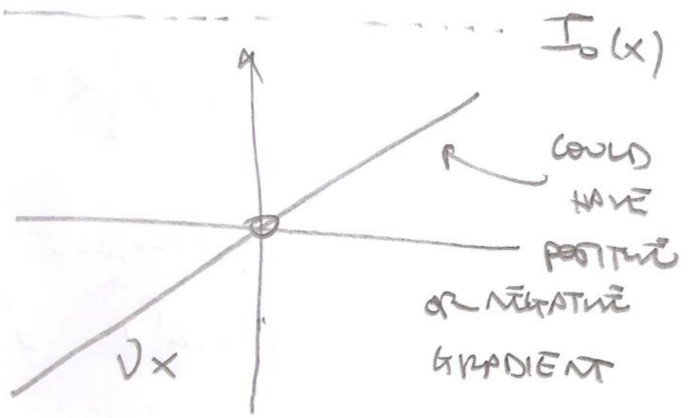
$$\text{minimize} \quad f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) + \sum_{i=1}^{p} I_0(h_i(x))$$

ON THE OTHER HAND, THE DUAL FUNCTION COMES FROM:

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

SO WE SUBSTITUTE HARD BOUNDS WITH SOFT <u>LOWER</u> BOUNDS:





WE MAY THINK OF λ_i, ν_i AS EXPRESSING DISPLEASURE (OR PLEASURE) FOR VIOLATION OF BOUNDS

5.1.5 EXAMPLES

EXAMPLE 1: LEAST SQUARES

minimize: $x^T x$

subject to: $Ax = b$, $A \in \mathbb{R}^{p \times n}$

LAGRANGIAN $L(x, \nu) = x^T x + \nu^T (Ax - b)$, DOMAIN $\mathbb{R}^n \times \mathbb{R}^p$

DUAL FUNCTION:

$$g(\nu) = \inf_x \left[x^T x + \nu^T Ax - b \right]$$

CONVEX QUADRATIC FUNCTION

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \iff x = -\left(\tfrac{1}{2}\right) A^T \nu$$

(NOTE: $\nabla\left(\tfrac{1}{2} x^T P x\right) = Px, \quad \nabla q^T x = q$)

$$g(\nu) = L\left(-\left(\tfrac{1}{2}\right)A^T\nu, \nu\right) =$$

$$\nu^T A\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}A^T\nu\right) + \nu^T\left[A\left(-\tfrac{1}{2}\right)A^T\nu - b\right]$$

$$= \frac{1}{4}\nu^T A A^T \nu - \frac{1}{2}\nu^T A A^T \nu - \nu^T b = -\frac{1}{4}\nu^T A A^T \nu - \nu^T b$$

WHICH IS CONCAVE, AS EXPECTED, SINCE

$$x^T\left[-\frac{1}{4}AA^T\right]x = -\frac{1}{2}\left[\left[A^Tx\right]^T\left[A^Tx\right]\right] \leq 0 \quad \forall x$$

THE INEQUALITY SPECIFIES

$$-\left(\frac{1}{4}\right)\nu^T A A^T \nu - b^T\nu \leq \inf\left\{x^Tx \mid Ax = b\right\}$$

<u>EXAMPLE 2</u>: STANDARD FORM LP

$$\left\{\begin{array}{ll} \text{minimize:} & c^Tx \\ \text{subject to:} & Ax = b \\ & x \geq 0 \end{array}\right\}$$

INEQUALITY CONSTRAINT FUNCTIONS:

$f_i(x) = -x_i$

<u>LAGRANGIAN</u>:

$$L(x, \lambda, \nu) = c^Tx - \sum_{i=1}^{n}\lambda_i x_i + \nu^T\left[Ax - b\right]$$

$$= \left[c + \nu^T A - \lambda\right]x - b^T\nu$$

<u>DUAL FUNCTION</u>:

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = -b^T\nu + \inf\left[c + \nu^T A - \lambda\right]x$$

SO

$$g(\lambda, \nu) = \begin{cases} -b^T\nu, & A^T\nu - \lambda + c = 0, \\ -\infty, & \text{OTHERWISE.} \end{cases}$$

THEREFORE, WHEN $A^T \nu - \lambda + c = 0$, $\lambda \geq 0$,

$-b^T \nu$ IS A LOWER BOUND OF THE ORIGINAL PROBLEM.

<u>EXAMPLE 3</u>: TWO-WAY PARTITIONING PROBLEM.

$$\left\{ \begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \dots, n \end{array} \right\}$$

<u>INTERPRETATION</u>: $x^T W x = \sum_{1 \leq i, j \leq n} x_i x_j W_{ij}$.

WE NEED TO BREAK PEOPLE $i = 1, \dots, N$ IN TWO GROUPS. WHEN i, j ARE TOGETHER, WE HAVE LOST W_{ij}. WHEN THEY ARE IN DIFFERENT GROUPS, WE HAVE LOST $-W_{ij}$.

$$L(x, \nu) = x^T W x + \sum_{i=1}^{n} \nu_i (x_i^2 - 1)$$

$$= x^T \left[W + \operatorname{diag}(\nu) \right] x - \mathbb{1}^T \nu$$

$$g(\nu) = \inf_x \; x^T \left[W + \operatorname{diag}(\nu) \right] x - \mathbb{1}^T \nu$$

$$= \left\{ \begin{array}{ll} -\mathbb{1}^T \nu, & W + \operatorname{diag}(\nu) \succeq 0 \\ -\infty, & \text{OTHERWISE} \end{array} \right.$$

THIS IS A LOWER BOUND TO OPTIMAL. FOR EXAMPLE, WE SET $\nu = -\lambda_{min}(W) \mathbb{1}$, IN WHICH CASE $W + \operatorname{diag}(\nu) \succeq 0$.

INDEED: $x^T \left[W - \lambda_{min}(W) I \right] x = x^T W x - \lambda_{min}(W) \|x\|_2^2$

$$\geq \lambda_{min}(W)\|x\|^2 - \lambda_{min}(W)\|x\|^2 \geq 0,$$ BECAUSE

OF BASIC PROPERTY OF EIGENVALUES

$$x^T W x \geq \lambda_{min} \|x\|_2^2.$$

(THEOREM: IF A SYMMETRIC,

$$\max_{\{x\,:\,\|x\|=1\}} x^T A x = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{min} = \min_{\{x\,:\,\|x\|=1\}} x^T A x$$)

SMALL DIVERSION TO THIRD CHAPTER:

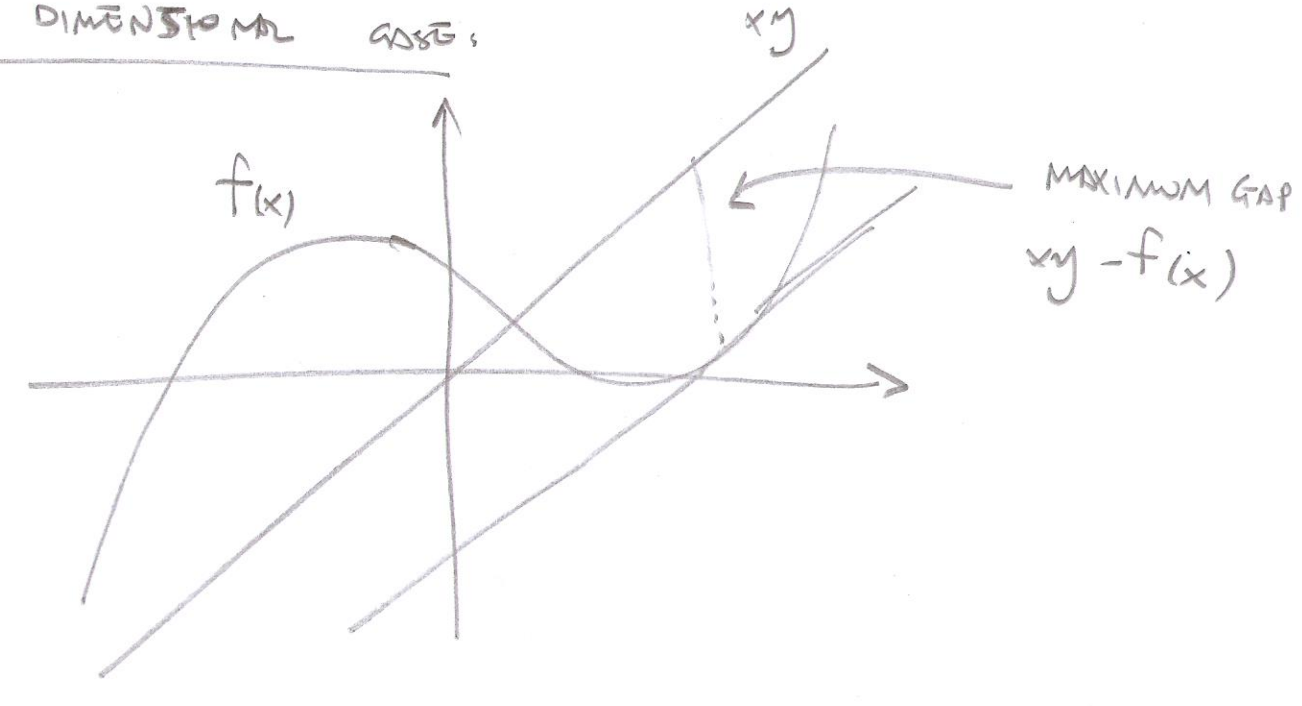
3.3 THE CONJUGATE FUNCTION

LET $f: \mathbb{R}^n \to \mathbb{R}$. THE CONJUGATE $f^*: \mathbb{R}^n \to \mathbb{R}$ IS DEFINED AS

$$f^*(y) = \sup_{x \in \mathrm{dom} f} \left(y^T x - f(x) \right)$$

WITH $\mathrm{dom} f^* = \{ y :$ THE ABOVE SUPREMUM IS FINITE $\}$

ONE DIMENSIONAL CASE:



OBSERVE THAT THE CONJUGATE IS CONVEX, BECAUSE IT IS THE SUPREMUM OF AFFINE FUNCTIONS

EXAMPLE: ONE-DIMENSIONAL CASES

1) $f(x) = ax+b$

$$f^*(y) = \sup_{x \in \mathrm{dom} f} (yx - ax - b),$$

WHICH IS ∞ UNLESS $y = a \Rightarrow$ $\underline{f^*(y) = -b, \ \mathrm{dom}\, f^* = \{a\}}$

2) $f(x) = -\log x$, $f^*(y) = \sup_{x \in \mathrm{dom} f} \{yx + \log x\}$.

SO WE NEED $y<0$, IN WHICH CASE $(yx + \log x)' = y + \frac{1}{x} = 0 \Leftrightarrow x = -\frac{1}{y} \Rightarrow$ $\boxed{f^*(y) = -1 - \log(-y), \ y<0}$

3) $f(x) = e^x$. $f^*(y) = \sup_x [yx - e^x]$. WE NEED $y \geq 0$, OTHERWISE $\sup = \infty$. IF $y=0$, $f^*(0) = 0$. IF $y>0$,

$(yx - e^x)' = y - e^x = 0 \Leftrightarrow x = \log y \Rightarrow$

$f^*(y) = y\log y - y \Rightarrow$ $\boxed{f^*(y) = y \log y - y, \ y \geq 0}$ $(0 \log 0 \triangleq 0)$

4) $f(x) = x\log x$, $x \geq 0$ $(0 \log 0 = 0)$.

$f^*(y) = \sup [yx - x\log x]$, WHICH EXISTS FOR ALL $y \in \mathbb{R}$

$(yx - x\log x)' = y - \log x - 1 = 0 \Leftrightarrow x = e^{y-1}$

$\Rightarrow \quad f^*(y) = y e^{y-1} - e^{y-1}(y-1) = e^{y-1}, \quad y \in \mathbb{R}$

5) $f(x) = \frac{1}{x}, \quad x > 0 \qquad f^*(y) = \sup\left(yx - \frac{1}{x}\right),$

WHICH IS BOUNDED ONLY FOR $y<0$. FOR $y=0$, $f^*(0) = 0$. FOR $y<0$, $\left(yx - \frac{1}{x}\right)' = y + \frac{1}{x^2} = 0 \Leftrightarrow$

$x^2 = -\frac{1}{y} \Leftrightarrow x = \sqrt{-\frac{1}{y}} \Rightarrow f^*(y) = y\sqrt{-\frac{1}{y}} - \left(\sqrt{-\frac{1}{y}}\right)^{-1} =$

$$\frac{-\sqrt{-y}\,\sqrt{-y}}{\sqrt{-y}} - \sqrt{-y} = -2(-y)^{\frac{1}{2}}, \quad y<0, \text{ so}$$

$$f^*(y) = -2\sqrt{-y}, \quad y \leq 0$$

EXAMPLE: STRICTLY CONVEX QUADRATIC FUNCTION

$f(x) = \frac{1}{2}x^T Q x, \quad Q \in S^n_{++}. \qquad f^*(y) = \sup_x \left(y^T x - \frac{1}{2}x^T Q x\right)$

WHICH ALWAYS HAS A MAXIMUM.

$$\nabla\left[y^T x - \frac{1}{2}x^T Q x\right] = y - Qx = 0 \Leftrightarrow \quad x = Q^{-1}y$$

$$\Rightarrow \quad f^*(y) = y^T Q^{-1} y - \frac{1}{2} y^T (Q^{-1})^T \cancel{Q\, Q^{-1}} y$$

$$= y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} y \Leftrightarrow \boxed{f^*(y) = \frac{1}{2} y^T Q^{-1} y}$$

EXAMPLE: (LOG-SUM-EXP)

$$f(x) = \log\left[\sum_{i=1}^{n} e^{x_i}\right]$$

(THIS FUNCTION INCREASES LINEARLY IN ALL PRINCIPAL DIRECTIONS AND SLOWER IN ALL OTHER ONES)

$\nabla\left[y^T x - f(x)\right] = 0 \Leftrightarrow$

$$\boxed{y_i = \frac{e^{x_i}}{\sum_{i=1}^{n} e^{x_i}}}$$

THIS CONDITION IS SUFFICIENT AND NECESSARY BECAUSE $y^T x - f(x)$ IS CONCAVE.

THEREFORE WE NEED $\mathbb{1}^T y = 1$, $y \geq 0$, OTHERWISE THERE CAN BE NO SOLUTION.

1) IF $y_i < 0$ FOR SOME i, SET $x_i = -t$, $x_j = 0$ $\forall j \neq i$,

AND WE GET

$$y^T x - f(x) = (-y_i)t - \log\left[(n-1) + e^{-t}\right] \to \infty.$$

2) IF $y \geq 0$, $\mathbb{1}^T y \neq 1$, WE SET $x = t\vec{\mathbb{1}}$,

$$y^T x - f(x) = t(y^T \mathbb{1}) - \log\left[n e^t\right] =$$

$$t\left(y^T \mathbb{1} - 1\right) - \log n,$$ WHICH CAN GO TO ∞ BY TAKING $t \to \pm\infty$.

3) IF $\mathbb{1}^T y = 1$, $y \succeq 0$ BUT SOME $y_i = 0$, THEN MAXIMUM IS ACHIEVED FOR $x_i \to -\infty$.

4) IF $\mathbb{1}^T y = 1$, $y > 0$, THERE IS A SOLUTION:

$$y^T x - \log\left(\sum_{i=1}^{n} e^{x_i}\right) = \sum y_i x_i - \log\left(\sum_{i=1}^{n} e^{x_i}\right)$$

$$= \sum_{i=1}^{n} y_i \log y_i + \underbrace{\sum_{i=1}^{n} y_i}_{=1} \log\left(\sum_{i=1}^{n} e^{x_i}\right) - \log\left(\sum_{i=1}^{n} e^{x_i}\right) \Rightarrow$$

$(*) \Leftrightarrow \log y_i = x_i - \log\left(\sum_{i=1}^{n} e^{x_i}\right)$

$$\boxed{f^*(y) = \sum_{i=1}^{n} y_i \log y_i \,, \quad y \succeq 0, \quad 1^T y = 1}$$

(SOME DETAILS ARE MISSING)

BASIC PROPERTIES

1) IF f CONVEX AND $\operatorname{epi} f$ IS CLOSED, THEN $f^{**} = f$.

2) CONJUGATE OF $g(x) = a f(x) + b$ IS

$$g^*(y) = a f^*\left(\frac{y}{a}\right) - b.$$

3) LET $A \in \mathbb{R}^{n \times n}$ NONSINGULAR, $b \in \mathbb{R}^n$. THEN CONJUGATE OF $g(x) = f(Ax + b)$ IS

$$g^*(y) = f^*(A^{-T} y) - b^T A^{-T} y.$$

4) SUM OF INDEPENDENT FUNCTIONS

IF $f(u, v) = f_1(u) + f_2(v)$, f_1, f_2 CONVEX,

THEN $f^*(w, z) = f_1^*(w) + f_2^*(z)$

PROOF OF STAVROS THAT DOES NOT NEED CONVEXITY (SO I AM CONFUSED):

$$f^*(w,z) = \sup_{u,v} \left[w^T u + z^T v - f_1(u) - f_2(v) \right]$$

$$= \sup_{u} \left[w^T u - f_1(u) \right] + \sup_{v} \left[z^T v - f_2(v) \right]$$

$$= f_1^*(w) + f_2^*(z)$$

BACK TO CHAPTER 5:

5.16 LAGRANGE DUAL FUNCTION AND THE CONJUGATE

CONSIDER THE PROBLEM
$$\left\{\begin{array}{ll} \text{minimize:} & f_0(x) \\ \text{subject to:} & Ax \preceq b \\ & Cx = d \end{array}\right\}$$

THE DUAL FUNCTION IS:

$$g(\lambda, \nu) = \inf_x \left[f_0(x) + \lambda^T [Ax - b] + \nu^T (Cx - d) \right]$$

$$= -b^T\lambda - d^T\nu + \inf_x \left[f_0(x) + \left(A^T\lambda + C^T\nu\right)^T x \right]$$

$$= -b^T\lambda - d^T\nu - \sup_x \left[-\left(A^T\lambda + C^T\nu\right)^T x - f_0(x) \right]$$

$$= -b^T\lambda - d^T\nu - f_0^*\left(-A^T\lambda + C^T\nu\right)$$

WITH $\mathrm{dom}\, g = \{(\lambda, \nu) : -A^T\lambda - C^T\nu \in \mathrm{dom}\, f_0^*\}$

EXAMPLE #1
$$\left\{\begin{array}{ll} \text{minimize:} & f_0(x) = \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to:} & Ax \preceq b \\ & 1^T x = 1 \end{array}\right\}$$

THE CONJUGATE OF $x \log x$ IS e^{y-1}, AND THE CONJUGATE OF SUM IS SUM OF CONJUGATES, SO

$$f_0^*(y) = \sum_{i=1}^{n} e^{y_i - 1}$$

ALSO, $\underset{\in \mathbb{R}^n}{C = 1^T}$, $\underset{\in \mathbb{R}}{d = 1}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

ALSO, LET
$$A = [a_1, a_2 \ldots a_n] \iff A^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$$

ALSO, $\nu \in \mathbb{R}$, $\lambda \in \mathbb{R}^m$

USING RESULT ABOVE:

$$g(\lambda,\nu) = -b^T\lambda - \nu - f_0^*\left(-A^T\lambda - \nu\mathbf{1}\right)$$

$$= -b^T\lambda - \nu - \sum_{i=1}^{n} e^{-a_i^T\lambda - \nu - 1}$$

$$= -b^T\lambda - \nu - e^{-\nu-1}\sum_{i=1}^{n} e^{-a_i^T\lambda}$$

5.2 THE LAGRANGE DUAL PROBLEM

WHAT IS THE BEST LOWER BOUND PROVIDED BY THE DUAL FUNCTION? $\Rightarrow$

$$\left\{\begin{array}{l}\text{maximize } g(\lambda,\nu)\\ \text{subject to } \lambda \succeq 0\end{array}\right\} \text{ LAGRANGE DUAL PROBLEM}$$

λ^*, ν^* ARE DUAL OPTIMAL or OPTIMAL LAGRANGE MULTIPLIERS. THIS PROBLEM IS ALWAYS CONVEX, IRRESPECTIVE OF PRIMAL ONE.

EXAMPLE #1: LAGRANGE DUAL of STANDARD FORM LP:

$$\left\{\begin{array}{ll}\text{minimize:} & c^Tx\\ \text{subject to:} & Ax=b\end{array}\right\}$$

WE HAVE SEEN:

$$g(\lambda,\nu) = \begin{cases}-b^T\nu, & A^T\nu - \lambda + c = 0,\\ -\infty, & \text{OTHERWISE},\end{cases}$$

SO DUAL PROBLEM IS:

$$\left\{\begin{array}{ll}\text{maximize} & g(\lambda,\nu)\\ \text{subject to} & \lambda \succeq 0\end{array}\right\} \Leftrightarrow \left\{\begin{array}{ll}\text{maximize:} & -b^T\nu,\\ \text{subject to:} & A^T\nu - \lambda + c = 0\\ & \lambda \succeq 0\end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{ll}\text{maximize} & -b^T\nu\\ \text{subject to:} & A^T\nu + c \succeq 0\end{array}\right\} \text{ LP IN INEQUALITY FORM!}$$

EXAMPLE #2: LAGRANGE DUAL OF INEQUALITY FORM LP

$$\left\{\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}\right\} \quad (1)$$

$$L(x,\lambda) = c^T x + \lambda^T (Ax - b) = -b^T \lambda + (A^T \lambda + c)^T x$$

$$g(\lambda) = \inf_x L(x,\lambda) = \begin{cases} -b^T \lambda & A^T \lambda + c = 0, \\ -\infty, & \text{OTHERWISE.} \end{cases}$$

THEREFORE, THE DUAL PROBLEM IS
$$\left\{\begin{array}{ll} \text{maximize} & -b^T \lambda \\ \text{subject to} & A^T \lambda + c = 0 \\ & \lambda \succeq 0 \end{array}\right\}$$

WHICH IS AN LP IN STANDARD FORM (2)

WE CAN ALSO SHOW THAT THE DUAL OF (2) IS (1).

5.2.2 WEAK DUALITY

LET d^* BE OPTIMAL OF DUAL PROBLEM

LET p^* BE OPTIMAL OF PRIMAL PROBLEM.

$d^* \le p^*$: WEAK DUALITY. (THIS ALSO HOLDS WHEN $p^* = -\infty$ OR $+\infty$, OR $d^* = \infty$)

$p^* - d^*$ = OPTIMAL DUALITY GAP.

BOUND ALLOWS US TO FIND HOW GOOD A SOLUTION WE ALREADY HAVE IS.

$d^* = p^* \Rightarrow$ STRONG DUALITY

WHEN DOES STRONG DUALITY HOLD?

NOT ALWAYS! USUALLY WHEN ORIGINAL PROBLEM IS CONVEX.

A CONDITION THAT ENSURES THIS IS CALLED CONSTRAINT QUALIFICATION.

SLATER'S THEOREM: DUALITY GAP IS ZERO IF

$$\exists x \in \operatorname{relint} D \text{ SUCH THAT } \left\{ \begin{array}{l} f_i(x) < 0, \; i=1,\dots,m, \\ Ax = b \end{array} \right\}$$

AND THE PROBLEM IS CONVEX. THE ABOVE CONDITION MEANS THAT THE POINT x IS STRICTLY FEASIBLE.

REFINEMENT: DUALITY GAP IS ZERO EVEN IF STRICT FEASIBILITY DOES NOT HOLD FOR AFFINE INEQUALITY CONSTRAINTS: IF $f_1, \dots f_k$ ARE AFFINE, THEN STRONG DUALITY HOLDS IF $\exists x \in \operatorname{relint} D$ WITH

$$f_i(x) \leq 0, \; i = 1, \dots, k,$$

$$f_i(x) < 0, \; i = k+1, \dots, m, \qquad Ax = b$$

NOTES: 1) IF $\operatorname{dom} f_0$ IS OPEN AND WE HAVE ONLY LINEAR INEQUALITY AND EQUALITY CONSTRAINTS, THEN DUALITY GAP IS ZERO, BY SLATER'S CONDITION.

2) SLATER'S CONDITION ALSO IMPLIES THAT DUAL OPTIMAL VALUE IS ATTAINED, WHEN $d^* > -\infty$, I.E. $\exists$ DUAL FEASIBLE (λ^*, ν^*): $g(\lambda^*, \nu^*) = d^* = p^*$