

**ΟΙΚΟΝΟΜΙΚΟ
ΠΑΝΕΠΙΣΤΗΜΙΟ
ΑΘΗΝΩΝ**



ATHENS UNIVERSITY
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M.Sc. Program in Data Science

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Optimization Techniques

Linear Programming – The Simplex Method

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The Simplex Method

- Designed by Dantzig (1947)
 - One of the most important algorithms of the 20th century
 - An algorithm that behaves extremely well in practice despite its exponential complexity in worst case
 - The design of the algorithm and the quest for better algorithms also contributed to building a rich theory around linear programming



Polyhedra

- Simplex is trying to optimize a linear function over a polyhedron
- **Definition:** In \mathbb{R}^n , a polyhedron is defined by a set of linear inequalities on n variables

$$P = \{x: Ax \leq b\}$$

– Where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and A is an $m \times n$ matrix

- We will usually consider polyhedra in the form

$$P = \{x: Ax \leq b, x \geq 0\}$$

- A polyhedron is
 - **Infeasible**, if its feasible region is empty
 - **Bounded**, if there exists M , such that for every x in the feasible region, $\|x\|_2 \leq M$
 - **Unbounded**, if it is not bounded

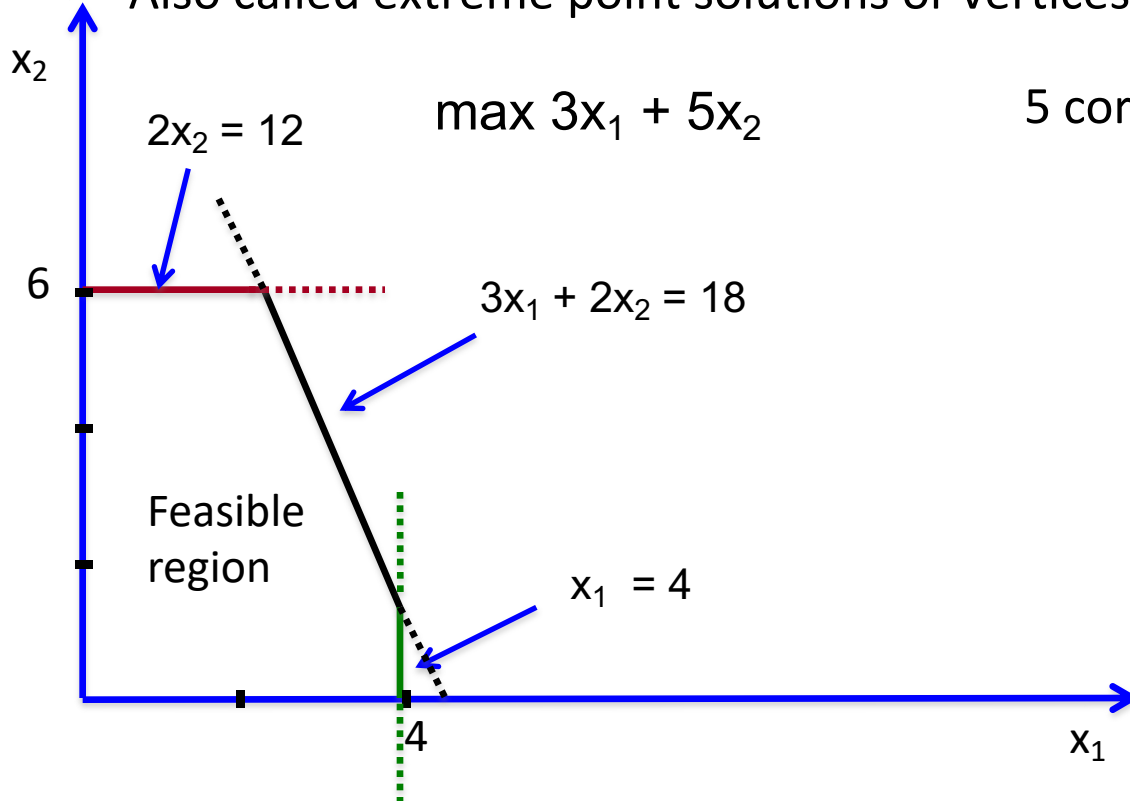
A Geometric Interpretation

- Simplex is an algebraic procedure
- However, it is important to understand its geometric motivation
- Assume the polyhedron is non-empty and bounded
 - Then, an optimal solution always exists for any linear objective function
 - A bounded polyhedron is also called polytope
- To illustrate the geometry of simplex, we will use Example 2 from Lecture 1 as a representative example in 2 dimensions

A Geometric Interpretation

Example 2: A polytope in \mathbb{R}^2

- **Constraint boundaries:** correspond to the 5 sides of the polygon
- **Corner point feasible (CPF) solutions:** points at the intersection of constraint boundaries
- Also called extreme point solutions or vertices of the polytope



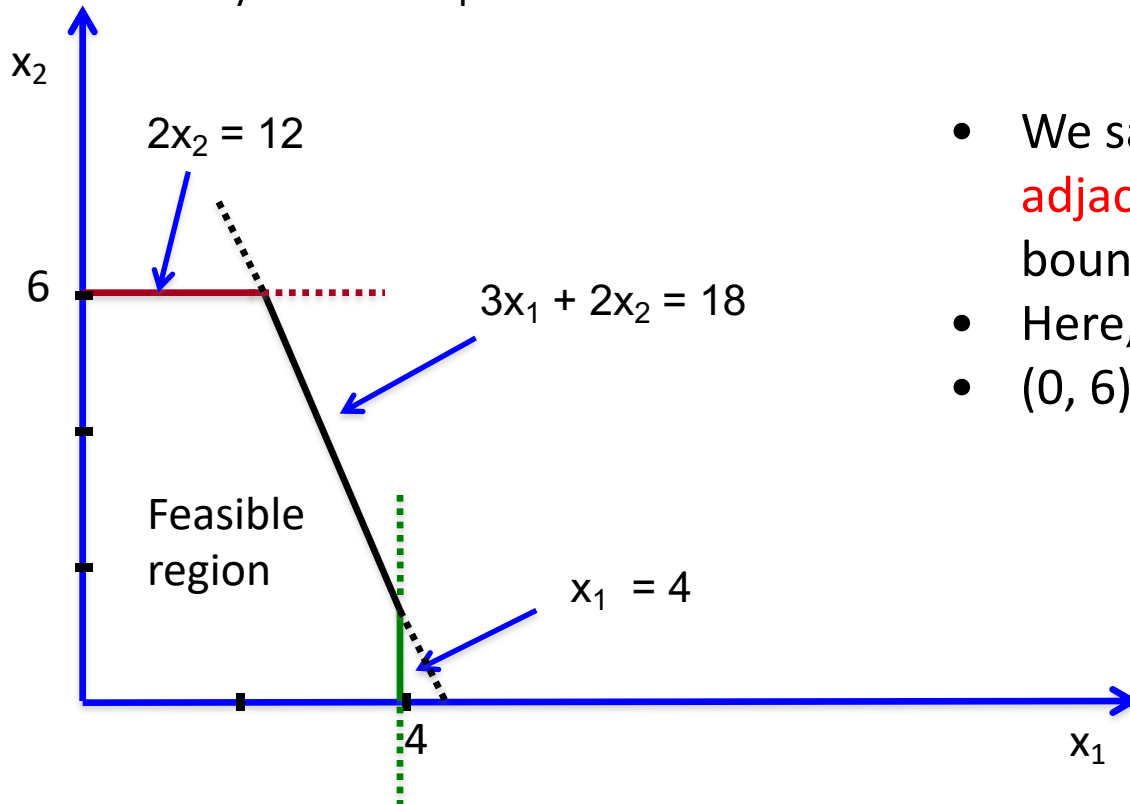
5 corner point solutions here:

- $(0, 0)$,
- $(0, 6)$,
- $(2, 6)$,
- $(4, 3)$,
- $(4, 0)$

A Geometric Interpretation

Example 2: A polytope in \mathbb{R}^2

- Each corner point solution lies at the intersection of 2 constraint boundaries
- In 2 dimensions: how do we find each CPF solution?
 - System of 2 equations in 2 variables



- We say 2 corner point solutions are **adjacent** if they share 1 constraint boundary
- Here, $(0,0)$ and $(0,6)$ are adjacent,
- $(0,6)$ and $(2,6)$ are also adjacent

A Geometric Interpretation

Generalization to n dimensions:

In a polyhedron with n variables,

- a CPF solution is the intersection of n constraint boundaries
- How do we identify them?
 - system of n equations in n variables
 - **Attention:** make sure we have first removed “redundant” constraints
 - i.e., constraints that can be implied by linear combinations of the others (otherwise the system will not have a unique solution)
 - each group of **n linearly independent constraints** of the polyhedron yields a distinct CPF solution
- Two CPF solutions are adjacent if they share $n-1$ constraint boundaries

A Geometric Interpretation

Why are we interested in the notion of adjacent solutions?

Optimality test for linear programs:

Consider a LP with at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better, according to the objective function, then it must be an optimal solution.

- Hence, **local optimality** \Rightarrow **global optimality**
- Extremely important property
 - Also generalizes to continuous, convex functions (to be discussed in next lectures)
- In our example: $(2, 6)$ is an optimal solution
 - $(2, 6)$ is adjacent to $(0, 6)$ and $(4, 3)$
 - None of these achieve a better value for the objective function

A Geometric Interpretation

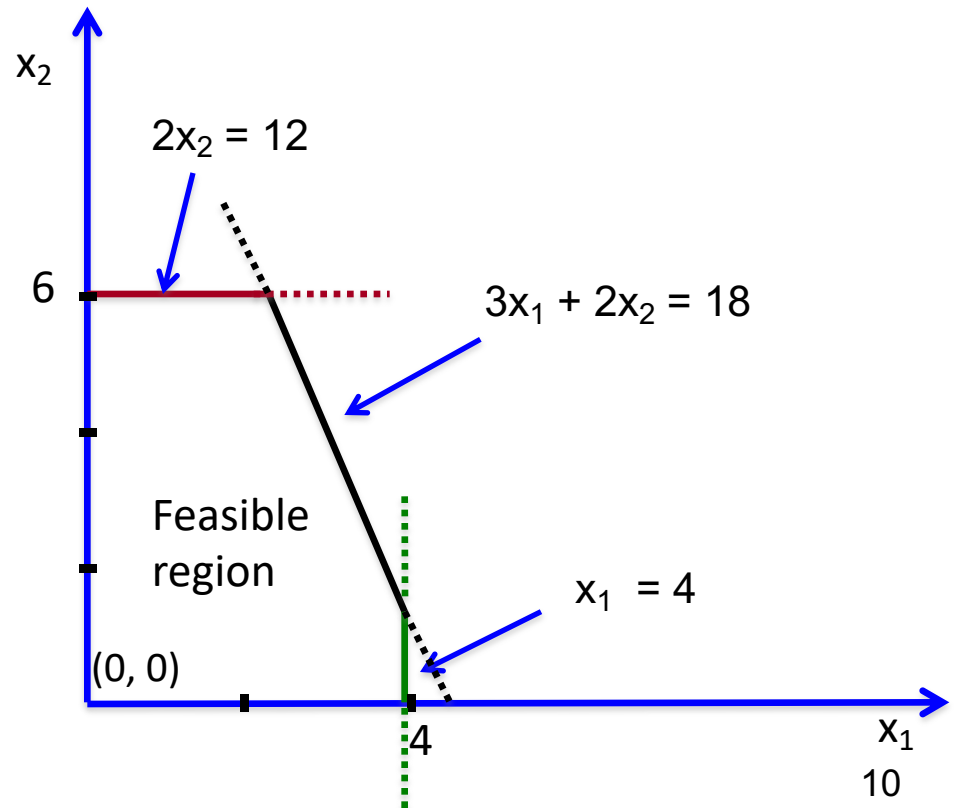
Outline of the simplex method from a geometric viewpoint

- **Initialization:** Choose an initial CPF solution
 - Usually we set all variables to 0
- **Main iteration loop:**
 - Apply the optimality test to the current CPF solution
 - If it is optimal stop,
 - else move to an adjacent solution that achieves **the highest rate of increase** in the objective function

A Geometric Interpretation

Solving Example 2 with the simplex method

- **Initialization:**
 - we choose $(0, 0)$ as the initial CPF solution
 - **Optimality test:** $(0, 0)$ is not an optimal solution, there are better adjacent solutions

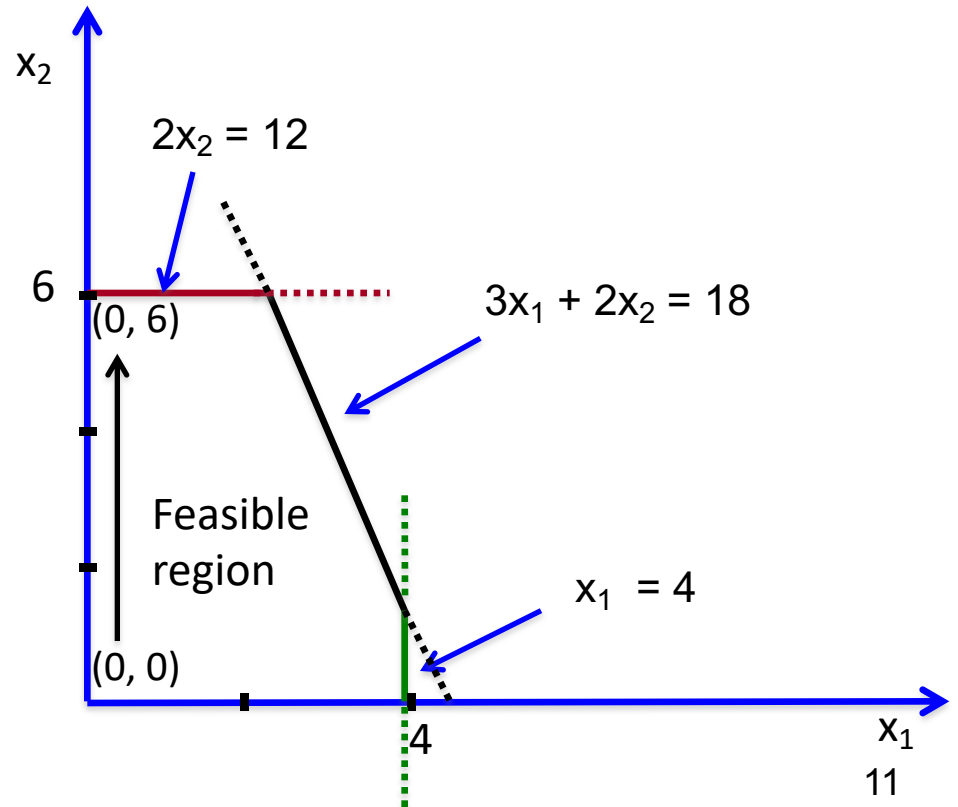


A Geometric Interpretation

Solving Example 2 with the simplex method

- **Iteration 1:**

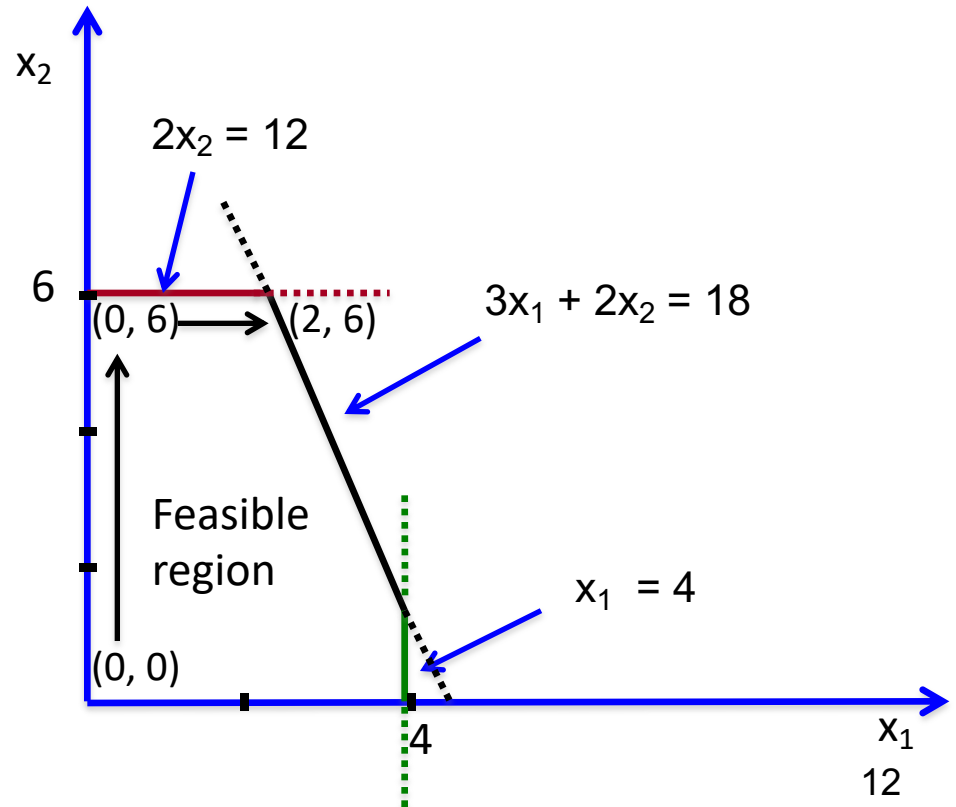
- Move from $(0, 0)$ to an adjacent solution
- How do we pick one?
- Choose the direction that increases the objective function at a faster rate
- Recall: $Z = 3x_1 + 5x_2$
- Hence moving along the x_2 axis is better, stopping at $(0, 6)$
- **Optimality test:** $(0, 6)$ is not optimal



A Geometric Interpretation

Solving Example 2 with the simplex method

- **Iteration 2:**
 - Move from $(0, 6)$ to a better adjacent solution
 - Moving back is not making things better
 - Hence, only choice to move to $(2, 6)$
 - **Optimality test:** $(2, 6)$ is better than $(0, 6)$ and $(4, 3)$, therefore, it is an optimal solution



A Geometric Interpretation

Basic features of simplex

- It only examines CPF solutions
 - It is guaranteed that there always exists an optimal CPF solution
- Initialization: Whenever feasible, take $(0, 0, \dots, 0)$
 - Nonnegativity constraints satisfied
 - What if the remaining constraints are violated? To be discussed again soon
- Picking the next CPF solution to visit:
 - Looking only at adjacent solutions can be easily implemented
 - The method only looks at the rate of increase in the objective function
 - Greedy local choice: we choose the direction with the best increase and stop at the adjacent solution in that direction

From Geometry to Algebra

Q: How can we implement all these steps in an automated algebraic manner for any polyhedron with n variables?

- We can use the geometric viewpoint only up to $n=3$ variables
- For $n > 3$, we need a translation into precise algebraic instructions

Setting up the simplex method

First step: Transform the standard form into a system of linear equations

- Conversion of inequality constraints into equalities by introducing *slack variables*
- For example: consider the inequality $x_1 \leq 4$ of Example 2
- We can define the slack variable: $x_3 = 4 - x_1$
- The constraint then is converted as:

$$x_1 \leq 4 \Rightarrow x_1 + x_3 = 4$$

- We can do this for all inequality constraints

Setting up the simplex method

Conversion of Example 2

- Need 3 slack variables: x_3, x_4, x_5

Original standard form

$$\max. Z = 3x_1 + 5x_2$$

s. t.:

$$x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1 \geq 0, x_2 \geq 0$$



Augmented form

$$\max. Z = 3x_1 + 5x_2$$

s. t.:

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_i \geq 0, i = 1, \dots, 5$$

Algebraically, more convenient to work with the augmented form

Setting up the simplex method

Some terminology:

- **Augmented solution:** simply a solution for the original variables augmented by the slack variables
 - For the feasible solution $(3, 2)$, the augmented solution is $(3, 2, 1, 8, 5)$
- **Basic Feasible (BF) solution:** an augmented CPF solution
 - $(0, 6)$ is a CPF solution in the original problem
 - $(0, 6, 4, 0, 6)$ is the corresponding BF solution
 - From a BF solution, we can get back the CPF solution by simply omitting the slack variables
- Understanding how BF solutions look like:
 - Example 2: 5 variables in total and 3 constraints
 - Hence, 2 degrees of freedom
 - If we set “arbitrary values” to 2 variables, then we can solve a linear system for the rest
 - In simplex: “arbitrary value” = 0

Setting up the simplex method

- For every BF solution:
 - We separate the variables into **basic** and **nonbasic** variables
 - Number of basic variables = m = number of constraints (excluding the nonnegativity constraints)
 - Number of nonbasic variables = n
 - Nonbasic variables are set to 0
 - Basic variables are then computed by solving the system of m linear equalities
 - The set of basic variables is referred to as the “**basis**” of the BF solution
- In our example:
 - $(0, 0, 4, 12, 18)$ is a BF solution
 - Nonbasic variables: x_1, x_2 , both set to 0
 - Basis = $\{x_3, x_4, x_5\}$
 - The values of the basis can be obtained by the constraints, after substituting $x_1 = x_2 = 0$

Setting up the simplex method

- Checking adjacency of two BF solutions
 - We could check if the corresponding CPF solutions are adjacent
 - **Easier way:** Two BF solutions are adjacent if their bases differ only in one variable (which means that all but one of their nonbasic variables are also the same)
- Illustration:
 - Adjacent CPF solutions: $(0, 0)$ and $(0, 6)$
 - Corresponding BF solutions: $S_1 = (0, 0, 4, 12, 18)$ and $S_2 = (0, 6, 4, 0, 6)$
 - In S_1 , nonbasic variables = $\{x_1, x_2\}$, basis = $\{x_3, x_4, x_5\}$
 - In S_2 , nonbasic variables = $\{x_1, x_4\}$, basis = $\{x_2, x_3, x_5\}$
 - Going from S_1 to S_2 , variable x_2 switches from nonbasic to basic and variable x_4 leaves the basis
- **Hence:** very simple way of moving from one adjacent solution to another

Setting up the simplex method

Final step before running simplex:

- It becomes convenient to also treat the objective function as another equality constraint

max. Z

s. t.:

$$Z - 3x_1 - 5x_2 = 0$$

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_i \geq 0, \quad i = 1, \dots, 5$$

- No need for a slack variable since we have equality to begin with
- We will not really treat Z as a new variable

Algebraic description of the simplex method

Initialization:

- We need to choose an initial BF solution
 - In our example, setting $x_1 = 0$ and $x_2 = 0$ is feasible
 - Augmented solution: $(0, 0, 4, 12, 18)$
 - Hence, initial basis = $\{x_3, x_4, x_5\}$, nonbasic variables: x_1, x_2
- Optimality test:
 - Initial value of the objective function: $Z = 0$
 - Recall $Z = 3x_1 + 5x_2$, expressed as a function of the nonbasic variables
 - Coefficient for each nonbasic variable: rate of improvement for Z , if that variable were to be increased.
 - Here the rates of improvement are positive, which means the current BF solution is not optimal
 - **Hence:** simple way to answer the optimality test

Algebraic description of the simplex method

Iteration 1:

- We need to determine the direction of movement towards an adjacent BF solution
 - Coefficient of x_2 in $Z >$ coefficient of x_1
 - We pick x_2 as the variable to increase
 - Variable x_2 will enter the basis (referred to as the **entering basic variable**)
- How much shall we increase x_2 ?
 - For as long as we do not violate the constraints!

$$(1) \quad x_1 + x_3 = 4 \quad \Rightarrow x_3 = 4$$

$$(2) \quad 2x_2 + x_4 = 12 \quad \Rightarrow x_4 = 12 - 2x_2$$

$$(3) \quad 3x_1 + 2x_2 + x_5 = 18 \quad \Rightarrow x_5 = 18 - 2x_2$$

And now use the nonnegativity constraints!

Algebraic description of the simplex method

Iteration 1:

$$(1) \quad x_1 + x_3 = 4 \quad \Rightarrow x_3 = 4$$

$$(2) \quad 2x_2 + x_4 = 12 \quad \Rightarrow x_4 = 12 - 2x_2$$

$$(3) \quad 3x_1 + 2x_2 + x_5 = 18 \quad \Rightarrow x_5 = 18 - 2x_2$$

$$x_3 \geq 0 \quad \Rightarrow \text{no upper bound on } x_2$$

$$x_4 \geq 0 \quad \Rightarrow 12 - 2x_2 \geq 0 \Rightarrow x_2 \leq 6$$

$$x_5 \geq 0 \quad \Rightarrow 18 - 2x_2 \geq 0 \Rightarrow x_2 \leq 9$$

- We pick the minimum value implied by the upper bounds
- Increasing x_2 beyond the value of 6 would result in an infeasible solution
- Hence, we stop at $x_2 = 6$

Algebraic description of the simplex method

Iteration 1:

- Can we arrive at $x_2 = 6$ with a more automated way?
- **Minimum Ratio Test:**
 - For each constraint, divide the constant term by the coefficient of x_2
 - The minimum such ratio tells us how much to increase x_2

$$(1) \quad x_1 + x_3 = 4 \quad \Rightarrow \text{ratio} = 4/0 = +\infty \text{ (0 coefficient of } x_2)$$

$$(2) \quad 2x_2 + x_4 = 12 \quad \Rightarrow \text{ratio} = 12/2 = 6$$

$$(3) \quad 3x_1 + 2x_2 + x_5 = 18 \quad \Rightarrow \text{ratio} = 18/2 = 9$$

- Setting $x_2 = 6$ makes variable x_4 drop to 0
- x_4 is called the *leaving basic variable*

Algebraic description of the simplex method

Iteration 1:

- Summarize what we have done so far:

	Initial BF solution	New BF solution
Nonbasic variables	$x_1 = 0, x_2 = 0$	$x_1 = 0, x_4 = 0$
Basis	$x_3 = 4, x_4 = 12, x_5 = 18$	$x_2 = 6, x_3 = ?, x_5 = ?$

- Final step of Iteration 1:
 - Convert the system of equations according to the new basis
 - Express the objective function in terms of the new nonbasic variables
 - Compute the missing values in the new BF solution (for x_3 and x_5)

Algebraic description of the simplex method

Iteration 1:

Initial constraints

$$(0) \quad Z - 3x_1 - 5x_2 = 0$$

$$(1) \quad x_1 + x_3 = 4$$

$$(2) \quad 2x_2 + x_4 = 12$$

$$(3) \quad 3x_1 + 2x_2 + x_5 = 18$$

- We need x_2 to disappear from (0), (1) and (3)
- Start with row (2): row (2) / 2 $\Rightarrow x_2 + 1/2 x_4 = 6$
- We can then
 - **multiply** a row by a constant
 - **Add/subtract** multiples of a row to/from another row
 - For example: row (0) := row (0) + 5 · row (2)

Algebraic description of the simplex method

Iteration 1:

Final set of constraints at the end of the iteration

$$(0) \quad Z - 3x_1 + 5/2 x_4 = 30$$

$$(1) \quad x_1 + x_3 = 4$$

$$(2) \quad x_2 + 1/2 x_4 = 6$$

$$(3) \quad 3x_1 - x_4 + x_5 = 6$$

- Procedure for obtaining the new form of the constraints: the **Gauss-Jordan method**
- Hence, assignment of values in the new BF solution:
 - $x_1 = 0, x_4 = 0, x_2 = 6, x_3 = 4, x_5 = 6$
- **Optimality test:**
 - $Z = 30 + 3x_1 - 5/2x_4$, positive coefficient for $x_1 \Rightarrow$ not optimal
 - Hence, we need to move to an adjacent BF solution

Algebraic description of the simplex method

Iteration 2:

- Which variable should now enter the basis?
 - **Unique choice:** Coefficient of x_1 is the only positive coefficient in Z
 - Variable x_1 is now the new entering basic variable
- How much shall we increase x_1 ?
 - Apply the Minimum Ratio Test
 - Set $x_1 := 2$ due to equation (3)
- Which variable exits the basis?
 - Again, from the Minimum Ratio Test, x_5 will be the leaving variable
- New basis: $\{x_1, x_2, x_3\}$
 - Nonbasic variables: $x_4 = x_5 = 0$
 - New BF solution: $(2, 6, 2, 0, 0)$

Algebraic description of the simplex method

Iteration 2:

Substituting using the Gauss-Jordan method:

$$(0) \quad Z + 3/2 x_4 + x_5 = 36$$

$$(1) \quad x_3 + 1/3 x_4 - 1/3 x_5 = 2$$

$$(2) \quad x_2 + 1/2 x_4 = 6$$

$$(3) \quad x_1 - 1/3 x_4 + 1/3 x_5 = 2$$

- **Optimality test:**

- $Z = 36 - 3/2 x_4 - x_5$
- There is no direction of improvement, increasing x_4 or x_5 will decrease the objective function
- Current BF solution is optimal
- Solution of the original linear program: $x_1 = 2$, $x_2 = 6$ and $Z = 36$

Tabular form of the simplex method

- So far we have managed to transform our geometric intuition into an algebraic procedure
- Operations used pretty simple
- Nevertheless, we can make the process even more automated
- **Simplex tableau:** A tabular representation of the constraints and the current BF solution
- All we need to know: the basis and the coefficients in each row

Tabular form of the simplex method

Algebraic form vs tableau:

Let us revisit the initialization in our Example:

$$(0) \quad Z - 3x_1 - 5x_2 = 0$$

$$(1) \quad x_1 + x_3 = 4$$

$$(2) \quad 2x_2 + x_4 = 12$$

$$(3) \quad 3x_1 + 2x_2 + x_5 = 18$$

← Algebraic form

Corresponding tableau form

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

rows = number of constraints + 1

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

- For notational convenience: treat Z also as a basic variable
- **Optimality test in a tableau:**
 - We have reached an optimal solution when the coefficients in row (0) are all nonnegative

Tabular form of the simplex method

Basis	Coefficients						Right side	
	Z	x_1	x_2	x_3	x_4	x_5		
Z	1	-3	-5	0	0	0	0	
x_3	0	1	0	1	0	0	4	
x_4	0	0	2	0	1	0	12	$12/2 = 6$
x_5	0	3	2	0	0	1	18	$18/2 = 9$

Iteration 1:

- Which variable should enter the basis?
 - The nonbasic variable with the most negative coefficient in row (0), hence x_2
 - Column of x_2 : **pivot column**
- Minimum Ratio Test
 - How do we run it?
 - Information we need is the right side column and the column of x_2

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

$12/2 = 6$
 $18/2 = 9$

Iteration 1:

- Outcome of the Minimum Ratio Test
 - Minimum achieved at row of x_4
 - Leaving variable: x_4 , i.e., the basic variable corresponding to that row
 - Row of x_4 : the **pivot row**
 - Intersection of pivot row and pivot column: **pivot element** (=2 in this iteration)

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

$12/2 = 6$

$18/2 = 9$

Iteration 1:

- Final step: Gauss-Jordan method to get the new tableau
 - Divide first the pivot row by the pivot element
 - This makes the coefficient of x_2 equal to 1 in the pivot row
 - Then we can add/subtract appropriate multiples of the pivot row to the other rows (just as in the algebraic description of simplex)

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	0	0	5/2	0	30
x_3	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
x_5	0	3	0	0	-1	1	6

New tableau

End of Iteration 1:

- Optimality test:
 - There exists a negative coefficient in row (0)
 - Hence, we need to go to the next iteration

Tabular form of the simplex method

Basis	Coefficients						Right side	
	Z	x_1	x_2	x_3	x_4	x_5		
Z	1	-3	0	0	5/2	0	30	
x_3	0	1	0	1	0	0	4	$4/1 = 4$
x_2	0	0	1	0	1/2	0	6	
x_5	0	3	0	0	-1	1	6	$6/3 = 2$

Iteration 2:

- Which variable should enter the basis?
 - Only variable x_1 has a negative coefficient in row (0)
 - **Pivot column:** The column of x_1
- Minimum Ratio Test
 - Variable x_5 is the leaving variable
 - **Pivot row:** The row of x_5

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	0	0	0	3/2	1	36
x_3	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

New tableau

End of Iteration 2:

- Optimality test
 - No negative coefficient in the row of Z
 - Hence we stop at the current BF solution (2, 6, 2, 0, 0)
 - Optimal solution to the original problem: $x_1 = 2$, $x_2 = 6$

Summary: Geometric, algebraic and tableau form

We have seen 3 different ways of thinking about the same algorithm

- **Geometric view:**
 - This is how the algorithm was inspired
 - Useful only for 2 or 3 dimensions
- **Algebraic description**
 - More convenient for learning the logic of the algorithm esp. in higher dimensions
- **Tableau form**
 - Equivalent to the algebraic form in terms of operations performed
 - However, it organizes the data in a more compact form
 - Allows for better automation

Tie-breaking and other technical details

Some issues that may arise

- **During the execution:**
 - Many choices for the entering variable
 - Many choices for the leaving variable
 - No leaving variable
- **At initialization:**
 - Difficulty in finding an initial feasible solution to begin with
- **At termination:**
 - Multiple optimal solutions

Tie-breaking and other technical details

- **Many choices for the entering variable:**
 - No problem, make an arbitrary choice
 - An optimal solution will be reached eventually
 - Hard to know in advance which one is the best choice

Tie-breaking and other technical details

- **Many choices for the leaving variable:**

- This may cause problems
- All such variables will become 0 at the end of the iteration
- Hence, we will have some basic variables with a 0 value
- Such solutions are called *degenerate*
- They may not allow Z to increase in the next iteration
- The algorithm may get trapped in a loop where some variables enter and exit the basis repeatedly and Z gets stuck at the same value
- **Bland's rule:** If there are multiple candidate leaving variables, always choose the variable with the smallest index
- Also: rarely been observed in practice, almost safe to ignore this

Tie-breaking and other technical details

- **No leaving basic variable:**

- This means that the entering variable can be increased indefinitely
- The increase does not yield any negative values to the current basic variables
- In the tableau form: all coefficients in pivot column are negative or 0 (except first row)
- **Conclusion:** The problem is unbounded, optimal solution is $+\infty$
- Maybe a mistake has occurred in the initial formulation of the problem

Tie-breaking and other technical details

- **Multiple optimal solutions:**

- If there are multiple optimal solutions, there are at least 2 optimal CPF solutions
- Any convex combination of these CPF solutions is also an optimal solution
- In some problems we may only care to identify one optimal solution and stop
- If we care to find more optimal CPF solutions:
 - Run more iterations of simplex after we found the first optimal solution
 - Choose a nonbasic variable with zero coefficient in the row of Z as the entering variable
 - There exists such a variable whenever there are multiple optimal solutions

Tie-breaking and other technical details

- **Difficulty in finding an initial feasible solution:**
 - What if the all-0 solution is not feasible? How do we start simplex then?
 - This can happen when some coefficients b_i are negative
 - **Strategy:** Define an auxiliary problem so that
 - It is easy to find an initial feasible solution in the auxiliary problem
 - The optimal solution of the auxiliary can tell us whether there exists a feasible solution in our original problem
 - **2-phase simplex method:**
 - First run simplex on the auxiliary problem
 - See whether we can identify an initial basic feasible solution from the optimal solution of the auxiliary problem
 - If yes, run simplex on our original problem

Tie-breaking and other technical details

- **Difficulty in finding an initial feasible solution:**
 - There are various ways to define the auxiliary problem
 - Illustration:

$$\max Z = c^T x$$

s.t.

$$\sum_j a_{ij}x_j \leq b_i$$

$$x_i \geq 0, i = 1, \dots, n$$

\Rightarrow

$$\min 1^T y$$

s.t.

$$\sum_j a_{ij}x_j + y_i = b_i, i = 1, \dots, n$$

$$x_i \geq 0, y_i \geq 0, i = 1, \dots, n$$

- The auxiliary problem always has a feasible solution
 - Set original variables to 0, and y equal to b.
- The original problem has a feasible solution if and only if the optimal of the auxiliary is 0

Other variants in implementing Simplex

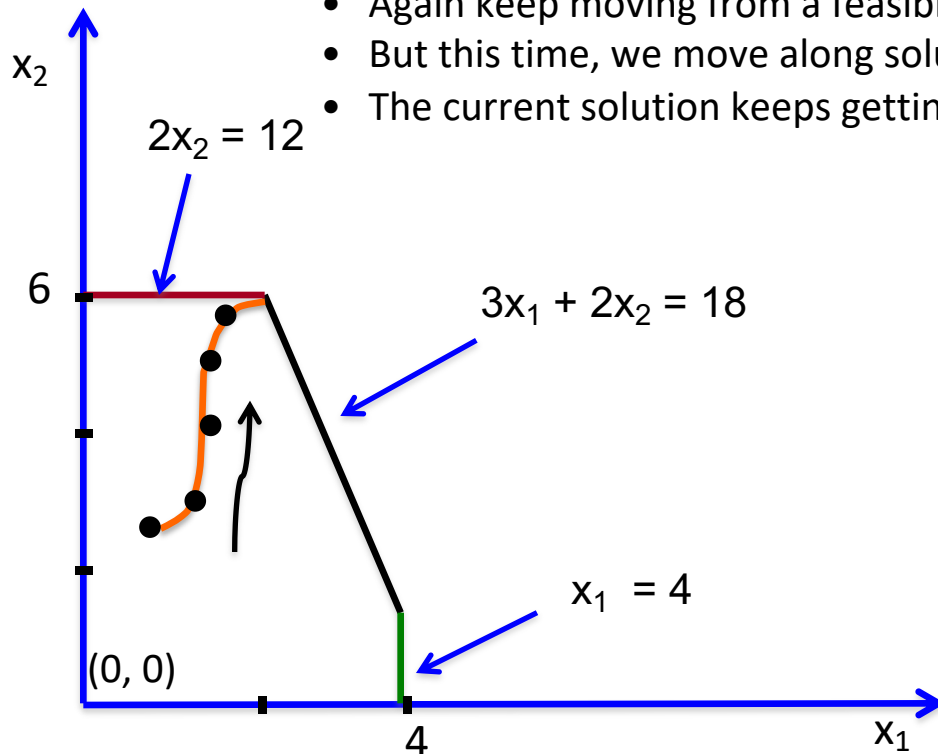
- The revised simplex method
 - Based on exploiting fast matrix operations
 - Each iteration requires solving 2 systems of linear equations
 - But these are not solved from scratch
 - Only small updates based on the solution from previous iteration
- The dual simplex method
 - Applying simplex to the dual linear program
 - But essentially working with the primal
 - Useful tool for sensitivity analysis
- Many other variations have also been suggested over the years...

Complexity of Simplex

- Extremely well-behaved in practice
- Empirically, number of iterations in simplex looks proportional to number of constraints, e.g., usually no more than $3m$
- Can we have a good theoretical upper bound on the number of iterations?
- **NO!** There are examples that need an exponential (2^n) number of iterations, discovered first by [Klee, Minty '72]
- Despite that, it is still one of the preferred algorithms for solving linear programs!

Other Algorithms

- **The ellipsoid method:** The first polynomial time algorithm
 - By [Kachiyan '79], however not well behaved in practice
- **Interior point methods:** also polynomial time algorithms
 - First conceived by Karmarkar [1984]
 - Main ideas:
 - Again keep moving from a feasible solution to a better one
 - But this time, we move along solutions in the interior of the polytope
 - The current solution keeps getting closer and closer to a vertex of the polytope



Other Algorithms

Types of interior point methods

- **Affine scaling algorithms**

- One of the simplest interior point algorithms
- Based on approximating polyhedra by “ellipsoids”
- Optimizing over ellipsoids in each iteration
- Non-linear problems but solvable with closed form solutions

- **Potential reduction algorithms**

- Do not measure progress by the increase in the objective function
- Instead use a non-linear potential function

- **Path following algorithms**

- Transforms the initial problem into an unconstrained problem (or a problem with equality constraints)
- Incorporates the inequality constraints “ $x_i \geq 0$ ” into the objective function (logarithmic barrier function)
- Solves the resulting non-linear problem with Newton’s method

Simplex vs Interior Point Algorithms

- **Comparisons**

- **In theory:** interior point methods are polynomial time algorithms (for any n and m), simplex may need exponential time
- **In practice:** average case complexity of simplex very low compared to worst case
- One iteration of interior point methods needs much more computation time than in simplex to decide the next feasible solution
- But: as the number of constraints increases, interior point methods do not need much more iterations
- Number of iterations in simplex may increase rapidly as we increase the number of variables and constraints
 - Interior point methods go through the internal part of the polytope
 - Adding more constraints reduces the feasible region, by adding more constraint boundaries
 - Hence, for problems with many thousands of constraints, interior point methods seem to be the best hope