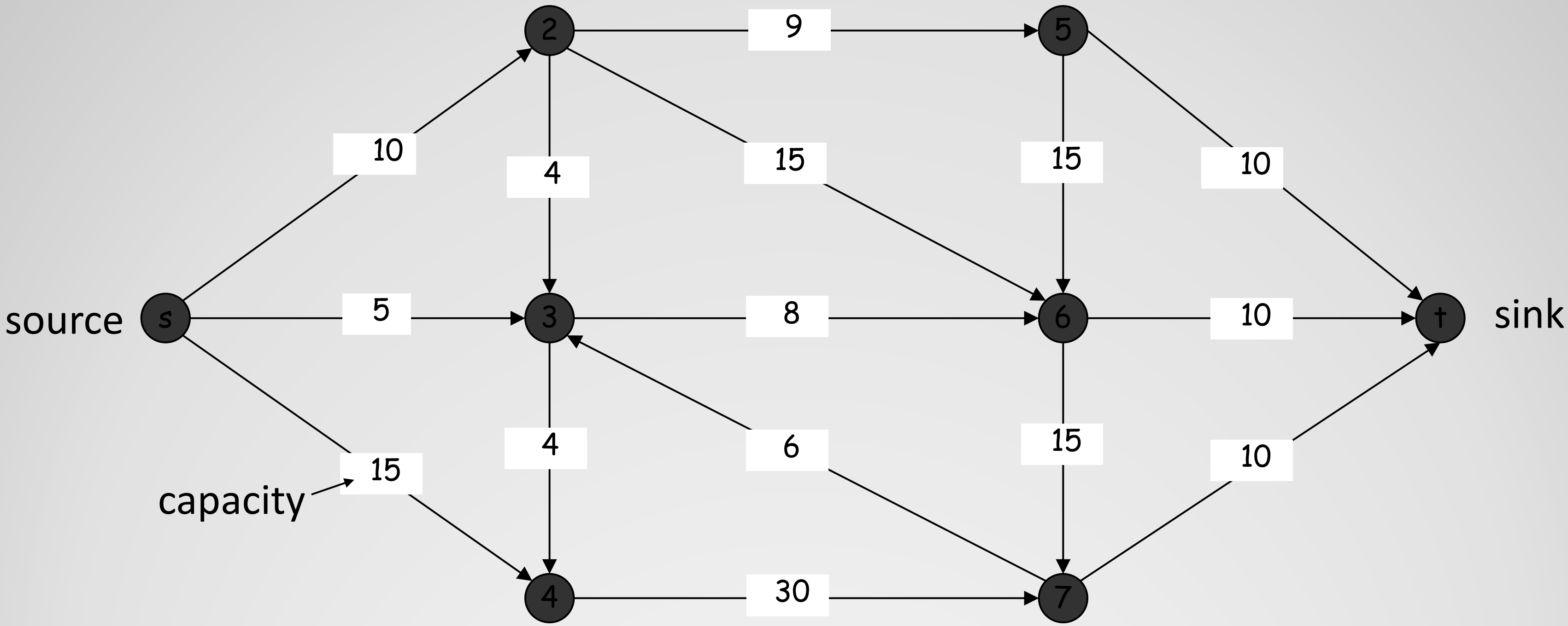
Maximum Flow and Minimum Cut

- Max flow and min cut.
  - Two very rich algorithmic problems.
  - Cornerstone problems in combinatorial optimization.
  - Beautiful mathematical duality.

- Nontrivial applications / reductions.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Bipartite matching.
  - Image segmentation.
  - Network connectivity.
  - Chemical Production
  - Network reliability.
  - Distributed computing.
  - Security of statistical data.
  - Many many more . . .

Flow network

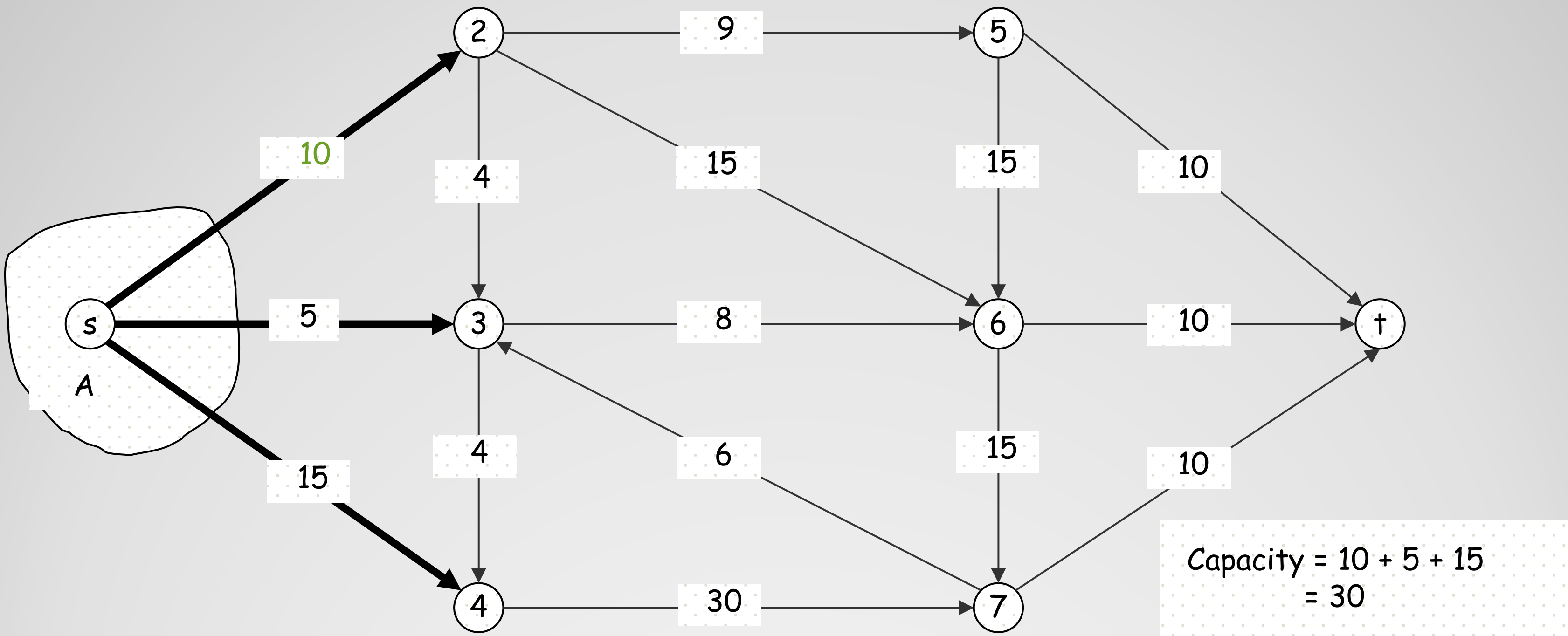
- Abstraction for material flowing through the edges.
- G = (V, E) = directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- c(e) = capacity of edge e.



Cuts

- Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

- Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



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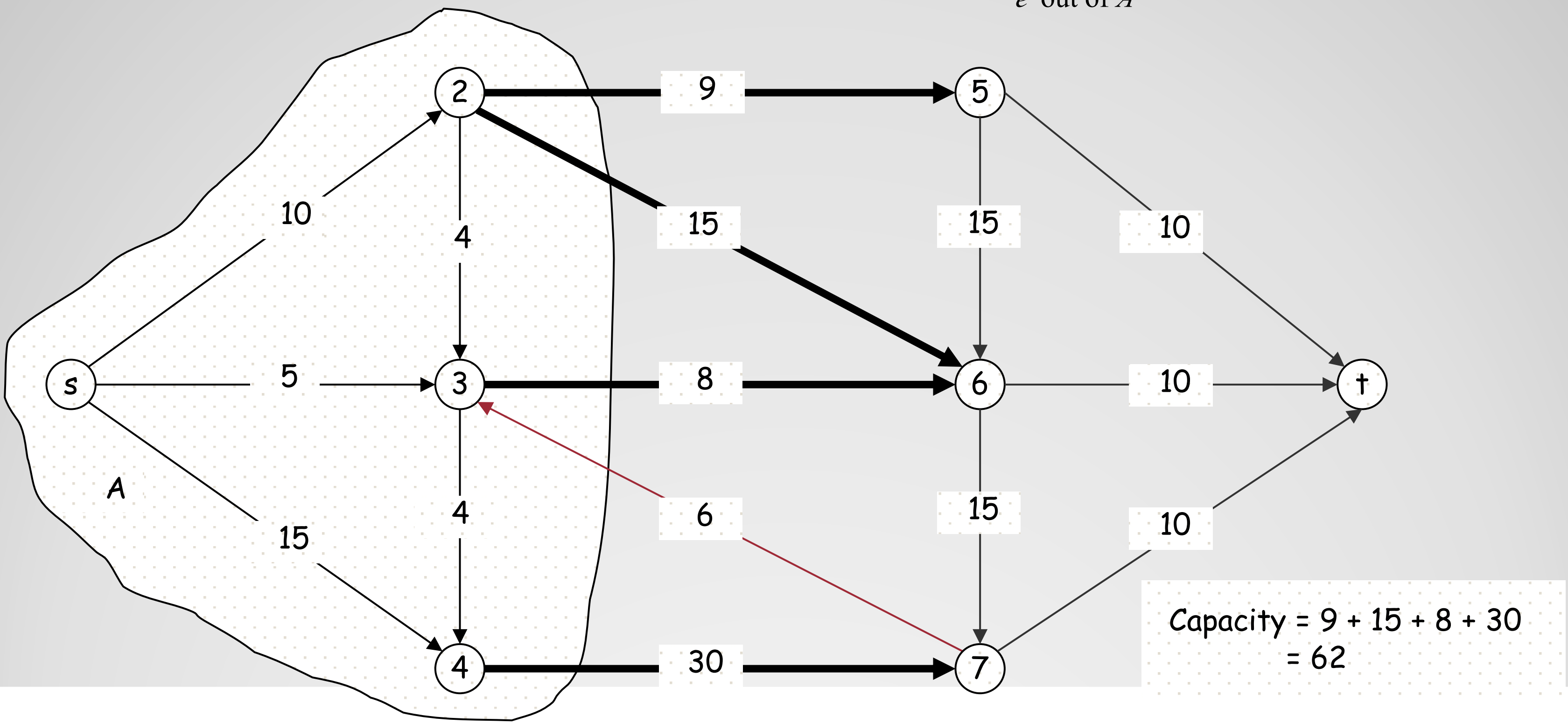
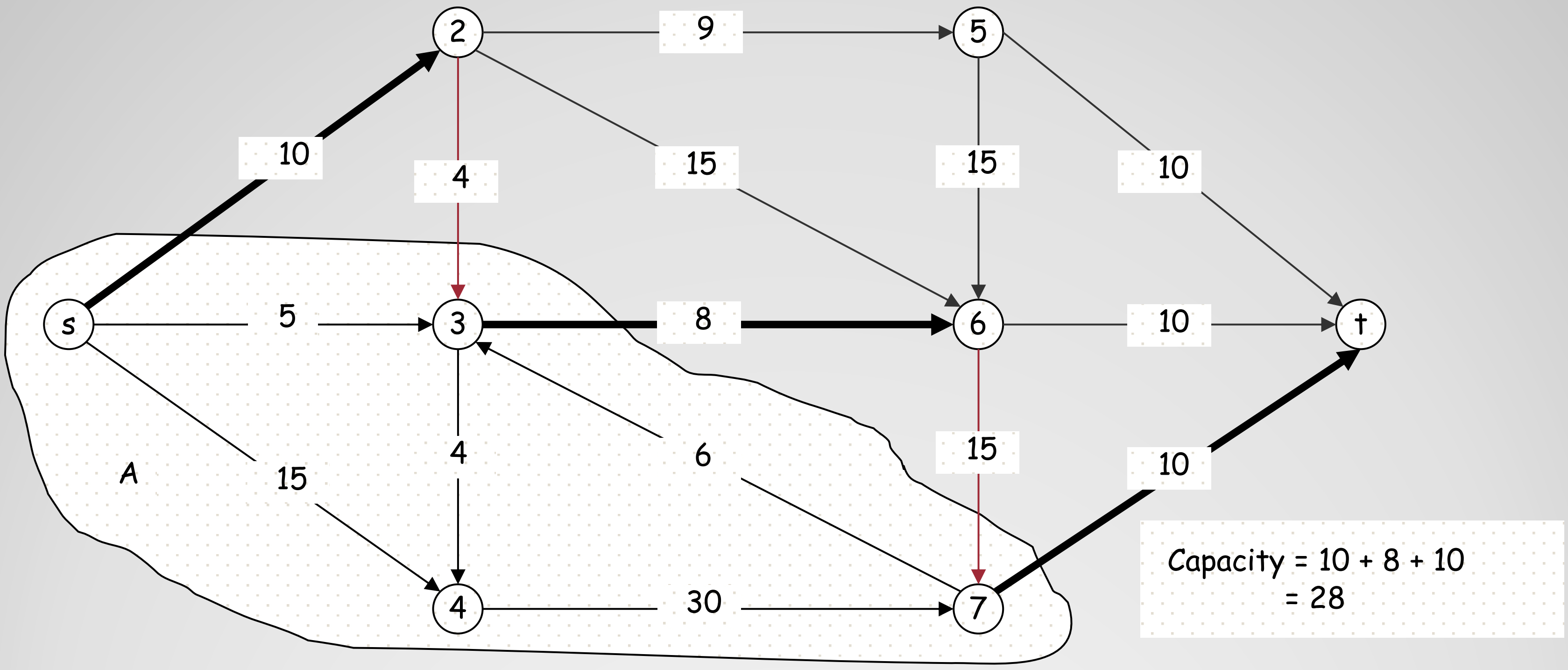


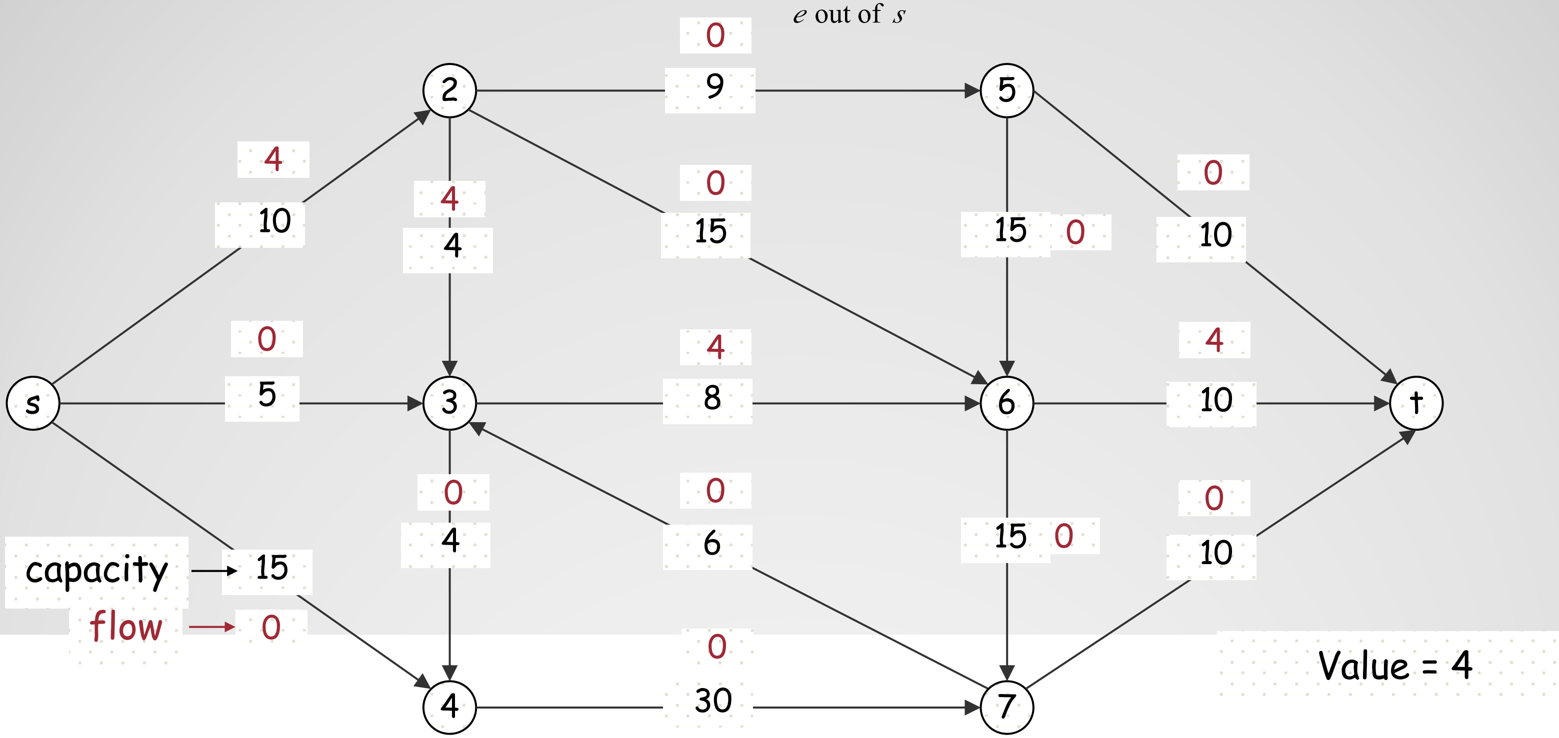
Minimum Cut Problem

- Min s-t cut problem. Find an s-t cut of minimum capacity.



Flows

- Def. An s-t flow is a function that satisfies:
  - For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
  - For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)
- Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



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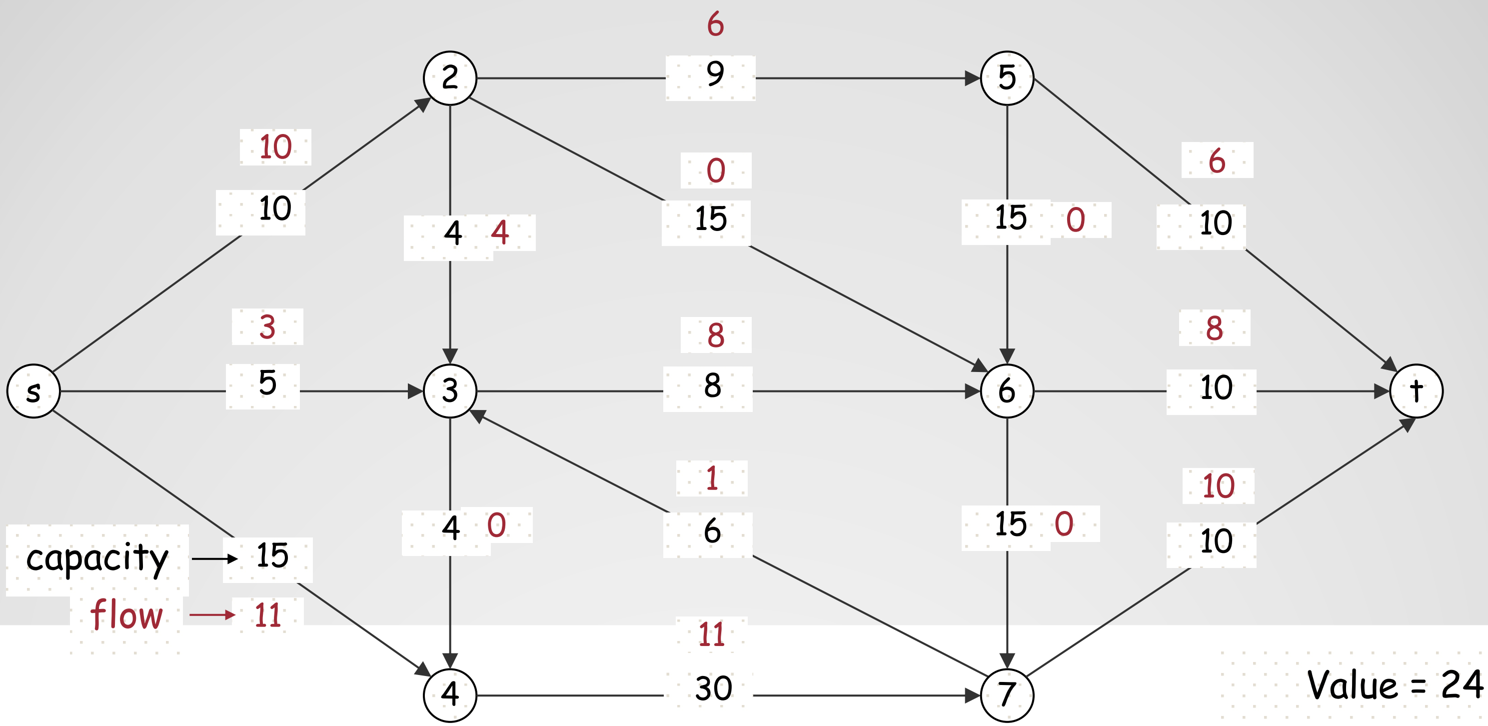
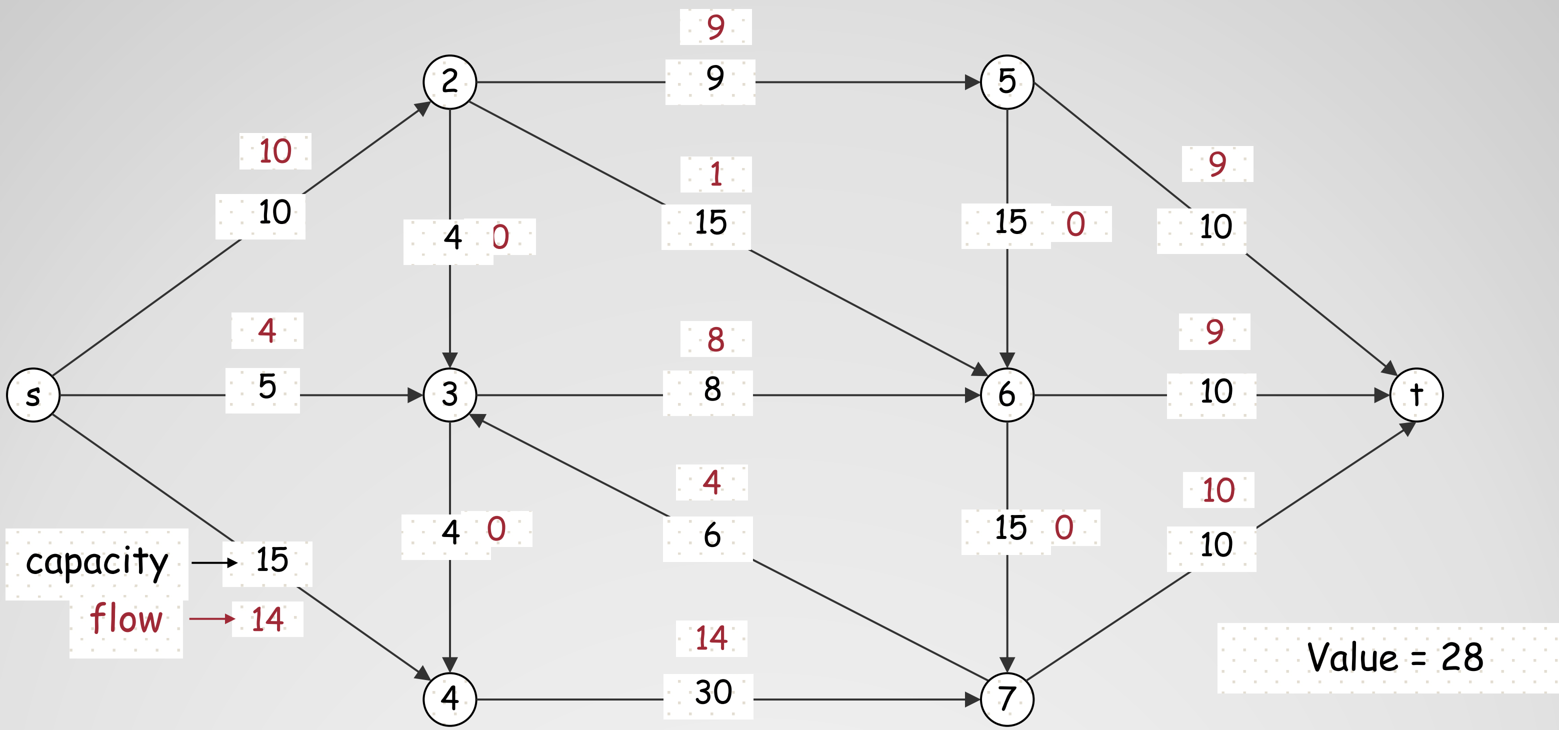


Maximum Flow Problem

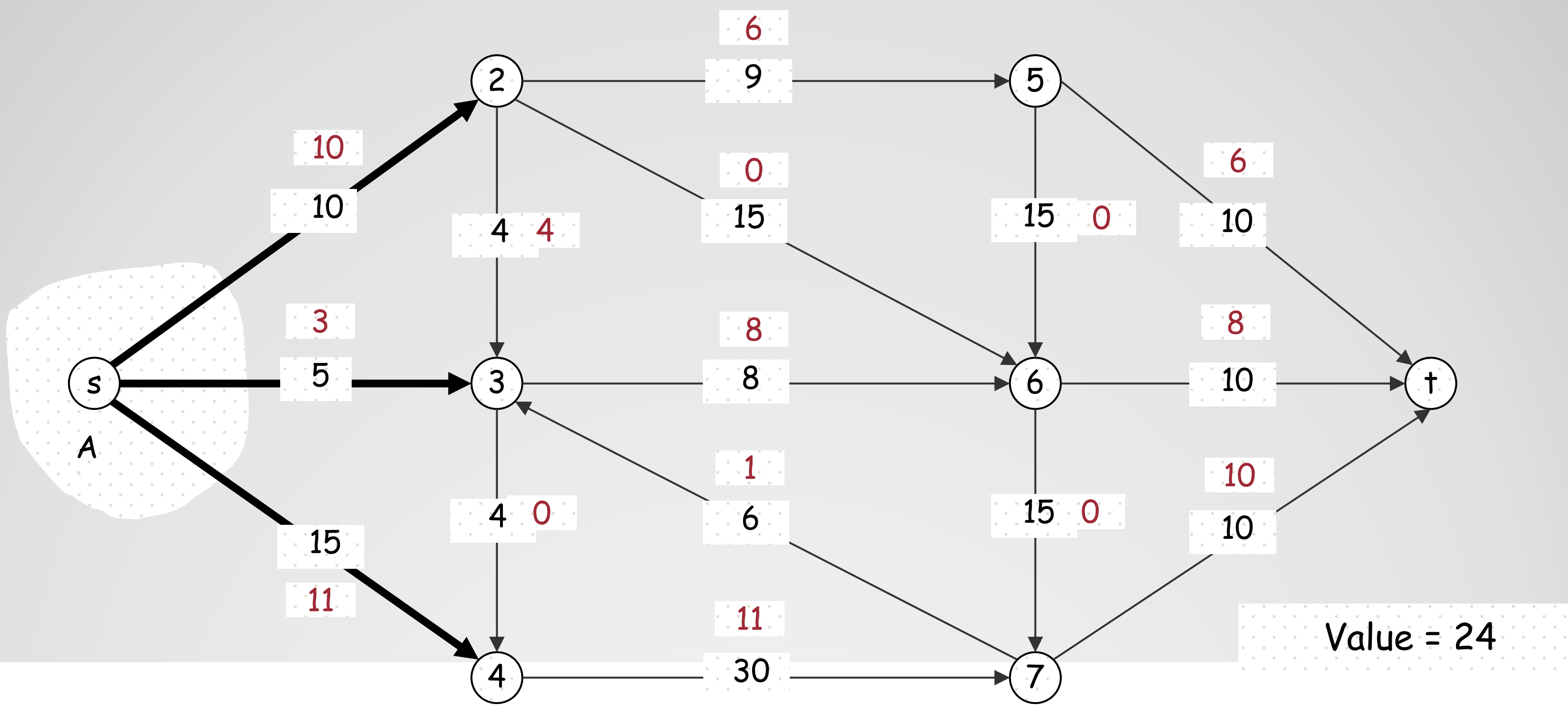
- Max flow problem. Find s-t flow of maximum value.



Flow and Cut Properties

P1: Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount of flow leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



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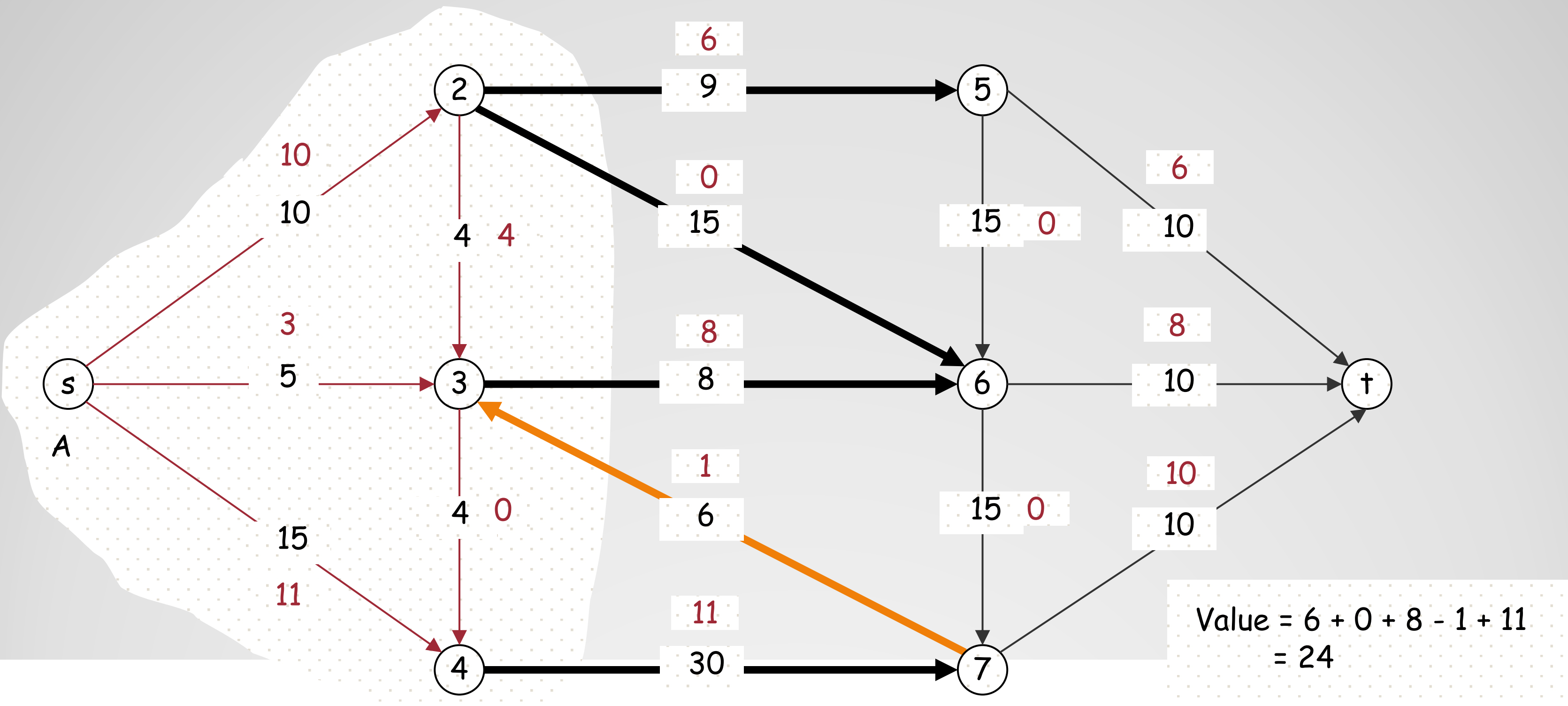

Flow and Cut Properties

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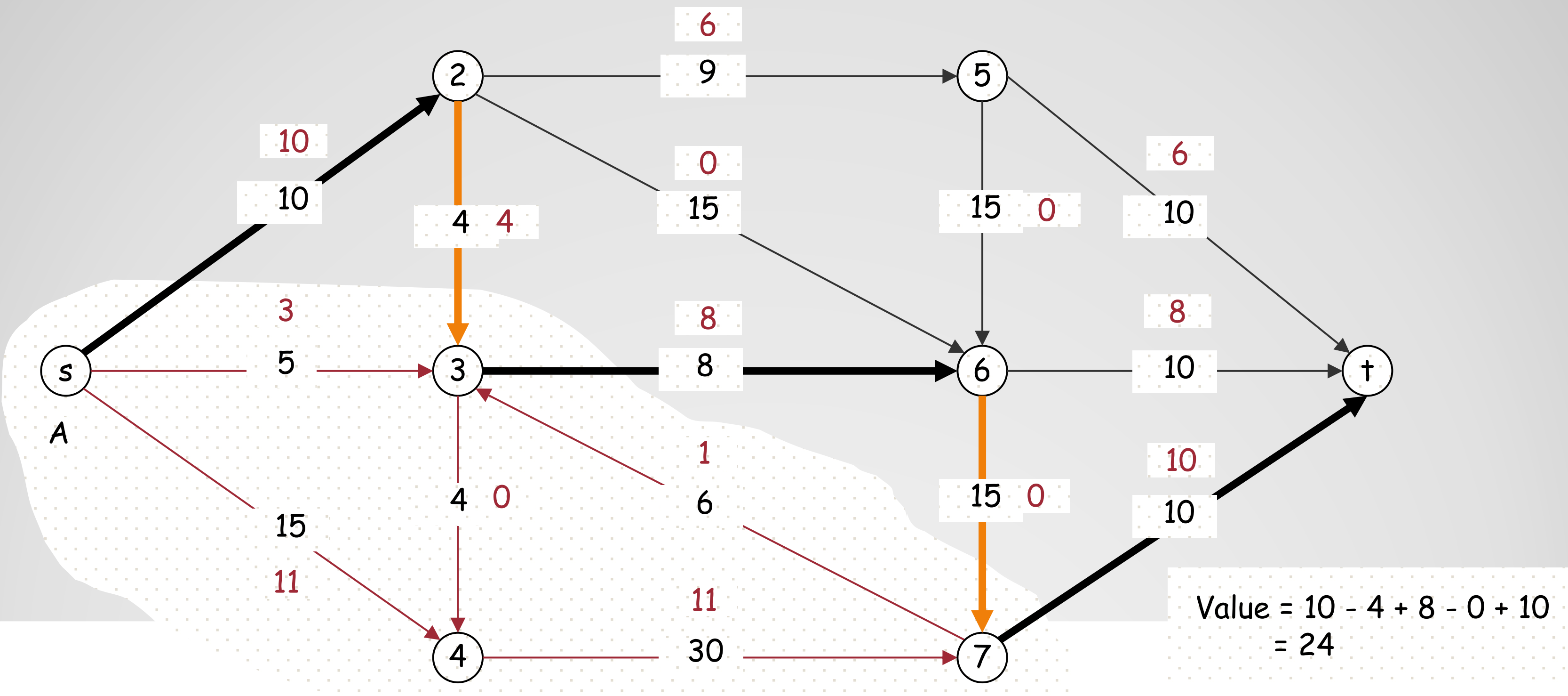

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Proof.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

By flow conservation, all terms except v = s are 0 $\longrightarrow$

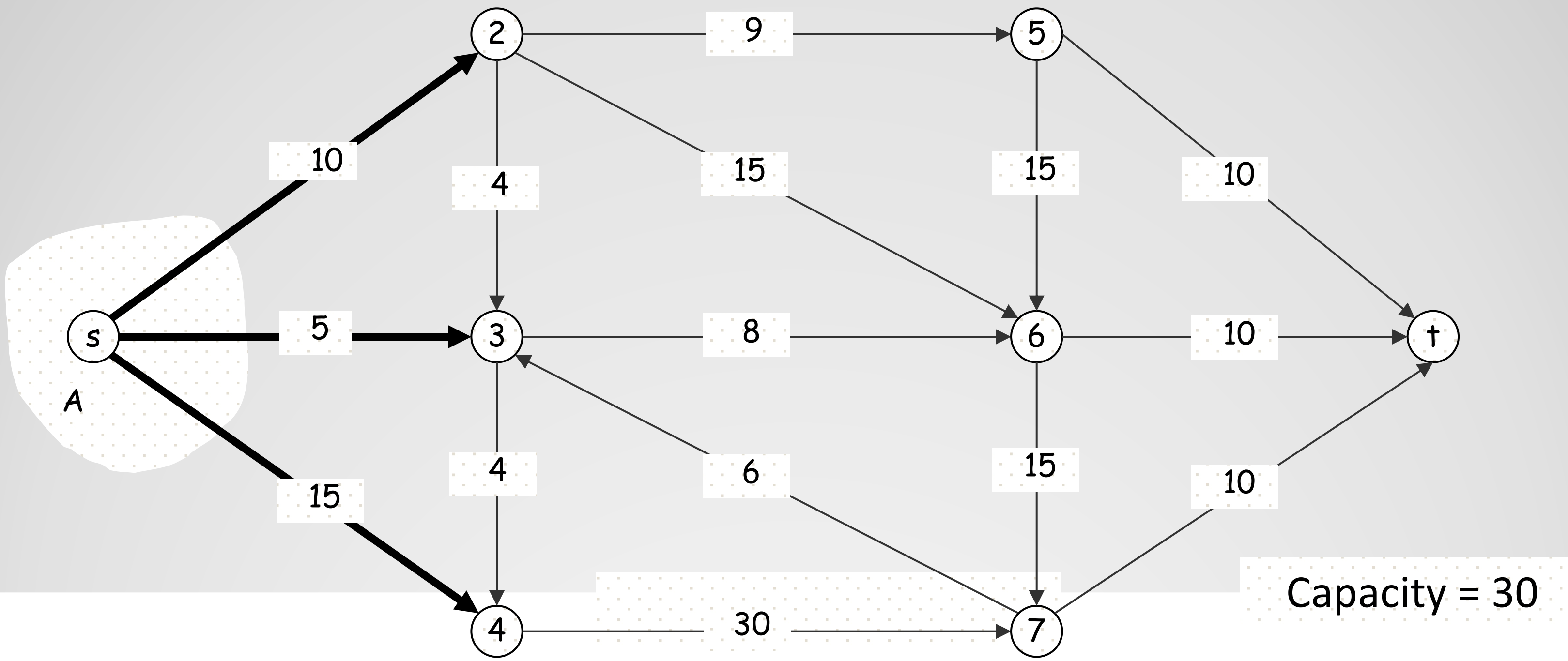
$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to v}} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e).$$

Flow and Cut Properties

***P2:** Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.*

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30



Flow and Cut Properties

P2: *Let f be any flow. Then, for any s-t cut (A, B) we have $v(f) \le cap(A, B)$.*

Proof.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&\le \sum_{e \text{ out of } A} f(e) \\
&\le \sum_{e \text{ out of } A} c(e) \\
&= \text{cap}(A, B)
\end{aligned}
$$

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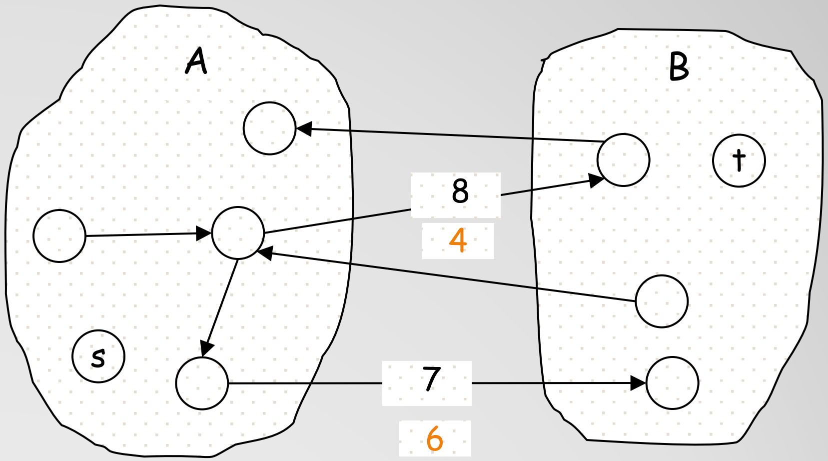

Certificate of Optimality

- *Max flow is at most equal to the capacity of the min cut (i.e., max flow is a lower bound to min cut)*
- *Let f be any flow, and let (A, B) be any cut.*
  - If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = Cut capacity = 28 $\Rightarrow$ Max flow value = 28 = Min cut capacity