ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

M.Sc. Program in Data Science
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Optimization Techniques

Linear Programming – Duality theory

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Outline

- Primal and Dual linear programs
  - Searching for upper bounds for the optimal solution
- The duality theorems
  - Weak and strong duality
- Complementary slackness optimality conditions
- Solving the dual using simplex
- Economic interpretation of dual variables
  - Sensitivity/post-optimality analysis

Finding lower bounds on the optimal solution

- Coming back to our illustrative example

$$
\begin{aligned}
\text{max. } & Z = 3x_1 + 5x_2 \\
\text{s. t.:} & \\
& x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

- Can we easily find a lower bound on the optimal solution?
- Q: Is the optimal solution at least 11?
  - Answer: yes because for example, $x_1 = 2$, $x_2 = 1$ is a feasible solution with a value of 11

Certificates for upper bounds

- In the opposite direction: Suppose we care for upper bounds

$$
\begin{aligned}
\text{max. } & Z = 3x_1 + 5x_2 \\
\text{s. t.:} & \\
& x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

- Can we certify that all feasible solutions are upper bounded by some value?
- How can someone convince us that Z ≤ 50?
- Q: Why should we care for upper bounds?
  - Recall it is a profit maximization problem, it could be useful to know in advance limitations on possible profit

Certificates for upper bounds

$$\begin{aligned}
\text{max. } & Z = 3x_1 + 5x_2 \\
\text{s. t.:} & \\
& x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}$$

A first attempt:

- Multiply the first inequality by 3: $3x_1 \leq 12$
- Multiply the second by 3: $6x_2 \leq 36$
- Add them up
- Hence, for every feasible solution:

$$Z = 3x_1 + 5x_2 \leq 3x_1 + 6x_2 \leq 48$$

Certificates for upper bounds

$$\begin{aligned} \text{max. } & Z = 3x_1 + 5x_2 \\ \text{s. t.:} & \\ & x_1 \qquad\quad\; \le 4 \\ & \qquad\;\; 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0,\; x_2 \ge 0 \end{aligned}$$

Even better:

- Multiply the second inequality by 2: $4x_2 \le 24$
- Multiply the third by 1: $3x_1 + 2x_2 \le 18$
- Add them up

$$Z = 3x_1 + 5x_2 \le 3x_1 + 6x_2 \le 42$$

- What is the best upper bound we can derive by such reasoning?

Certificates for upper bounds

General strategy:

- We try to construct linear combinations of the constraints
- We will do it parametrically
- Let y_i = multiplier of the i-th constraint
- We will not use the nonnegativity constraints

Constraints:

$(\ x_1 \leq 4\ )\ y_1$

$(\ 2x_2 \leq 12\ )\ y_2$

$(\ 3x_1 + 2x_2 \leq 18\ )\ y_3$

Add them up $\Rightarrow$

$$(y_1 + 3y_3)x_1 + (2y_2 + 2y_3)x_2 \leq 4y_1 + 12y_2 + 18y_3$$

Certificates for upper bounds

What information can we get from:

$(y_1 + 3y_3)x_1 + (2y_2 + 2y_3)x_2 \leq 4y_1 + 12y_2 + 18y_3$ (*)

Observation 1: We need that all y_i's are nonnegative

- Otherwise, the inequalities are reversed

Observation 2: In order for (*) to imply an upper bound for $Z(x) = 3x_1 + 5x_2$, we need that

$$3x_1 + 5x_2 \leq (y_1 + 3y_3)x_1 + (2y_2 + 2y_3)x_2$$

Hence we need to enforce that:

$y_1 + 3y_3 \geq 3$

$2y_2 + 2y_3 \geq 5$

Certificates for upper bounds

How can we get the best possible upper bound?
By solving the minimization problem:

$$\min W(y) = 4y_1 + 12y_2 + 18y_3$$

s.t.

$$y_1 + 3y_3 \geq 3$$

$$2y_2 + 2y_3 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

- This is yet another linear program
- Referred to as the *"dual"* of the original linear program
- Original program also referred to as the *"primal"* program

Primal and Dual Linear Programs

For every primal linear program, we can construct a unique dual linear program

$$\max Z(x) = 3x_1 + 5x_2$$
s.t.
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

$$\min W(y) = 4y_1 + 12y_2 + 18y_3$$
s.t.
$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$

- primal maximization LP $\Rightarrow$ dual minimization LP
- Number of variables in the dual = number of constraints in the primal
- Number of constraints in the dual = number of variables in the primal

Primal and Dual Linear Programs

General form of primal and dual programs

Both the primal and the dual are defined on the same set of parameters

Given:

- $c_1, c_2, \ldots, c_n$
- $b_1, b_2, \ldots, b_m$
- The constraint matrix $A = (a_{ij})$ with $1 \le i \le m$, $1 \le j \le n$,

Primal program

maximize $Z(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
&x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0
\end{aligned}
$$

Dual program

minimize $W(y) = b_1y_1 + b_2y_2 + \ldots + b_my_m$

subject to:

$$
\begin{aligned}
&a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1 \\
&a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2 \\
&\vdots \\
&a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n \\
&y_1 \ge 0, y_2 \ge 0, \ldots, y_m \ge 0
\end{aligned}
$$

Primal and Dual Linear Programs

More concisely:

Primal program

$$\max \; Z(x) = c^T \cdot x$$

s. t.:

$$A \cdot x \leq b$$

$$x \geq 0$$

Dual program

$$\min \; W(y) = b^T \cdot y$$

s. t.:

$$A^T \cdot y \geq c$$

$$y \geq 0$$

Claim: The dual of the dual program is the primal program!

- i.e., following the same approach of multiplying the dual constraints with variables, you get exactly the primal!

Primal and Dual Linear Programs

Concise tabular format:

Primal variables

Dual variables

	x_1	x_2	…	x_n	Right side
y_1	a_{11}	a_{12}		a_{1n}	$\leq b_1$
y_2	a_{21}	a_{22}		a_{2n}	$\leq b_2$
…					…
y_m	a_{m1}	a_{m2}		a_{mn}	$\leq b_m$
Right side	$\geq c_1$	$\geq c_2$	…	$\geq c_n$	

- Primal program: Read constraints along the rows
- Dual program: Read constraints along the columns

Primal and Dual Linear Programs

Coming back to our example

Primal program

$\max Z(x) = 3x_1 + 5x_2$

s.t.

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

Dual program

$\min W(y) = 4y_1 + 12y_2 + 18y_3$

s.t.

$y_1 + 3y_3 \geq 3$

$2y_2 + 2y_3 \geq 5$

$y_1, y_2, y_3 \geq 0$

- Optimal solution to the primal: We have seen it is 36 ($x_1 = 2$, $x_2 = 6$)
- Optimal solution to the dual: It is also 36 ($y_1 = 0$, $y_2 = 3/2$, $y_3 = 1$)

Is this a coincidence?

Duality theorems

The Weak Duality Theorem:

Consider a primal linear program and its corresponding dual program such that both have feasible solutions

- Let x be a feasible solution to the primal program with cost $Z(x) = c^T x$
- Let y be a feasible solution to the dual program with cost $W(y) = b^T y$

Then $Z(x) \leq W(y)$

Note: We were expecting that this should be the case

- We constructed the dual as an attempt to find upper bounds on the optimal solution of the primal

Proof of weak duality:

- Since y is a feasible solution of the dual, we have: $c \leq A^T \cdot y$
- Thus $c^T \cdot x \leq (A^T \cdot y)^T \cdot x = (y^T \cdot A) \cdot x = y^T \cdot (A \cdot x) \leq y^T \cdot b = b^T \cdot y = W(y)$

Duality theorems

In fact, we can have something stronger:

The Strong Duality Theorem:

For any pair of primal and dual linear programs,

- The primal program has an optimal solution if and only if the dual has an optimal solution
- If x^* and y^* are optimal solutions to the primal and dual respectively, then $Z(x^*) = W(y^*)$ i.e. $c^T \cdot x^* = b^T \cdot y^*$

Proof by using the weak duality theorem and exploiting further properties of the 2 programs

Duality theorems

Example:

Primal program

$$\max Z(x) = 4x_1 + x_2 + 5x_3 + 3x_4$$

s.t.

$$x_1 - x_2 - x_3 + 3x_4 \leq 1$$

$$5x_1 + x_2 + 3x_3 + 8x_4 \leq 55$$

$$-x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Dual program

$$\min W(y) = y_1 + 55y_2 + 3y_3$$

s.t.

$$y_1 + 5y_2 - y_3 \geq 4$$

$$-y_1 + y_2 + 2y_3 \geq 1$$

$$-y_1 + 3y_2 + 3y_3 \geq 5$$

$$3y_1 + 8y_2 - 5y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

Consider the feasible solutions: $x = (0, 14, 0, 5)$ and $y = (11, 0, 6)$

- $Z(x) = 29$
- $W(y) = 29$
- The duality theorems directly imply that these are optimal solutions!

Derivation of the dual LP

Suppose we have a primal LP not in standard form

- How can we construct the dual then?
- We can always bring the LP to standard form
- But there is no need to
- Suppose we have a maximization problem with inequality and equality constraints
- We can apply almost the same procedure
  - One dual variable per constraint
  - For equality constraints $\Rightarrow$ dual variable not needed to be nonnegative
  - For primal variables that are not constrained to be nonnegative $\Rightarrow$ corresponding dual constraint must be an equality constraint
  - Objective function formed as before

Derivation of the dual LP

Example: Find the dual of the following LP

$$\max Z(x) = 4x_1 + x_2 + 5x_3 + 3x_4$$

s.t.

$$x_1 + 2x_2 - x_3 + 3x_4 \leq 1$$

$$5x_1 + x_2 + 4x_3 + 8x_4 = 20$$

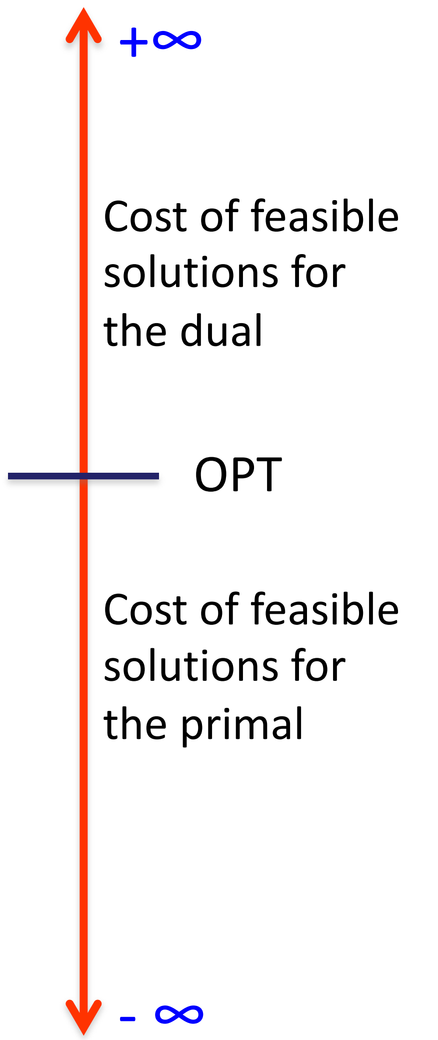
$$2x_1 + 5x_2 + 2x_3 - 5x_4 \leq 3$$

$$x_1, x_3 \geq 0$$

Consequences of the duality theorems

The following are the only possible situations that can occur:

- If the primal has feasible solutions and the feasible region is bounded, then both the primal and the dual have an optimal solution with the same value for their objective function
- If the primal is unbounded, then the dual is infeasible
- If the primal is infeasible, then
  - Either the dual is infeasible as well
  - Or the dual is unbounded

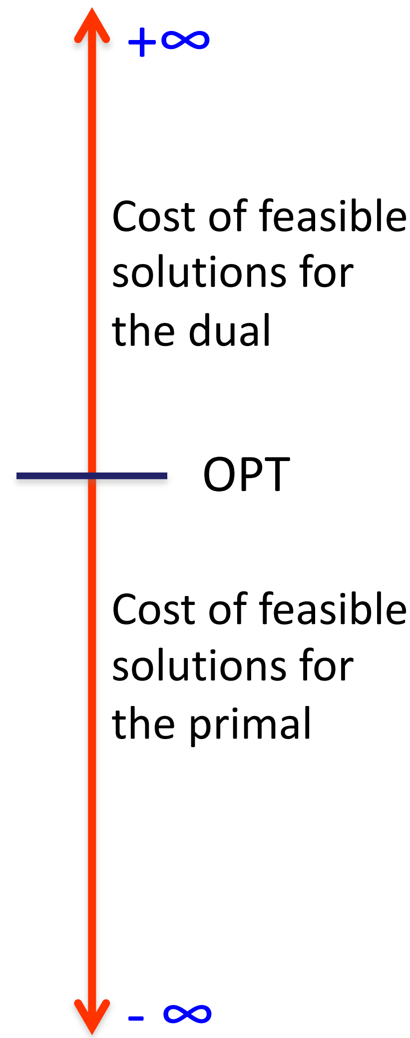


Consequences of the duality theorems

Primal

Dual

	Optimal solution	Unbounded	Infeasible
Optimal solution	✓	✗	✗
Unbounded	✗	✗	✓
Infeasible	✗	✓	✓



Consequences of the duality theorems

Example: Consider the following primal LP

Primal program

$\max Z(x) = x_1 + 2x_2$

s.t.

$x_1 + x_2 = 1$

$2x_1 + 2x_2 = 3$

Is the dual infeasible or unbounded?

The Complementary Slackness Conditions

- We can relate even further the optimal solutions of the 2 programs
- Note that every primal variable corresponds to a constraint in the dual
- Every dual variable corresponds to a constraint in the primal
- Consider a constraint of the primal, e.g. $3x_1 + 2x_2 \leq 18$
- Given a feasible solution, we say that a constraint is *tight* or *binding* if it is satisfied with equality
- Recall that at a corner point optimal solution we will have some tight constraints (by the definition of corner point solutions)
- Can we tell which constraints will be tight?
- The complementary slackness conditions relate the tightness of a constraint with the value of the corresponding dual variable

The Complementary Slackness Conditions

- Back to our example:

max $Z(x) = 3x_1 + 5x_2$	min $W(y) = 4y_1 + 12y_2 + 18y_3$
s.t.	s.t.
$x_1 \leq 4$	$y_1 + 3y_3 \geq 3$
$2x_2 \leq 12$	$2y_2 + 2y_3 \geq 5$
$3x_1 + 2x_2 \leq 18$	$y_1, y_2, y_3 \geq 0$
$x_1, x_2 \geq 0$	

- Primal optimal: $x_1 = 2$, $x_2 = 6$, Dual optimal: $y_1 = 0$, $y_2 = 3/2$, $y_3 = 1$

Observation on the primal constraints:

- $x_1 \leq 4$: loose, dual variable: $y_1 = 0$
- $2x_2 \leq 12$: tight, dual variable: $y_2 > 0$
- $3x_1 + 2x_2 \leq 18$: tight, dual variable: $y_3 > 0$

The Complementary Slackness Conditions

Theorem:

- Let x be a feasible solution of a primal program

$$\max \{ Z(x) = c^T \cdot x \mid A \cdot x \leq b, x \geq 0\}$$

- Let y be a feasible solution of the corresponding dual program

$$\min \{ W(y) = b^T \cdot y \mid A^T \cdot y \geq c, y \geq 0\}$$

- Let A_i := i-th row of A, and A^j := j-th column, for i=1,...,m, j=1,...,n

Then x and y are optimal solutions to the primal and the dual respectively if and only if

- For every j = 1,...,n, either $x_j = 0$ or $(A^j)^T \cdot y = c_j$ i.e., $x_j \cdot (c_j - (Aj)^T \cdot y) = 0$
- For every i = 1,...,m, either $y_i = 0$ or $A_i \cdot x = b_i$ i.e., $y_i \cdot (b_i - A_i \cdot x) = 0$

Interpretation: For feasible solutions x, y to be optimal for primal and dual

- If a primal constraint is not tight, the corresponding dual variable should be set to 0
- If a dual constraint is not tight, the corresponding primal variable should be set to 0

The Complementary Slackness Conditions

One more way to look at it:

- Recall that in the augmented form of the primal program, we added m slack variables
- For i = 1,..., m, $x_{n+i} = b_i - A_i \cdot x$
- We can also define slack variables in the dual program
- For j = 1,..., n, $y_{m+j} = c_j - A^j \cdot y$

The complementary slackness conditions can be written as:

- For every j = 1,...,n, $x_j \cdot y_{m+j} = 0$
- For every i = 1,...,m, $y_i \cdot x_{n+i} = 0$

Complementarity refers to the fact that in the augmented form,
either one variable of the primal or a corresponding dual variable has to be 0

The Complementary Slackness Conditions

Example of using the complementary slackness conditions

Primal program

$\max Z(x) = 4x_1 + x_2 + 5x_3 + 3x_4$

s.t.

$x_1 - x_2 - x_3 + 3x_4 \leq 1$

$5x_1 + x_2 + 3x_3 + 8x_4 \leq 55$

$-x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3$

$x_1, x_2, x_3, x_4 \geq 0$

Dual program

$\min W(y) = y_1 + 55y_2 + 3y_3$

s.t.

$y_1 + 5y_2 - y_3 \geq 4$

$-y_1 + y_2 + 2y_3 \geq 1$

$-y_1 + 3y_2 + 3y_3 \geq 5$

$3y_1 + 8y_2 - 5y_3 \geq 3$

$y_1, y_2, y_3 \geq 0$

- Suppose we solve first the dual and find: $y = (11, 0, 6)$
- Checking the dual constraints, and by complementary slackness we know that $x_1 = 0$, $x_3 = 0$
- Also since $y_1 > 0$, $y_3 > 0$, first and third primal constraints are tight
- Hence solving a system of 2 equations, we get $x = (0, 14, 0, 5)$

Back to the simplex algorithm

Can we solve the dual simultaneously with the primal?

- YES! The simplex algorithm solves both
- It suffices to look at the tableau form of simplex
- All the necessary information is located on row (0) of the tableau

A more detailed look at simplex:

- During all iterations, simplex maintains a primal feasible solution along with a candidate dual solution
- In all iterations before the last one, the candidate dual solution is infeasible and the primal is non-optimal
- In the last iteration, simplex finds both a primal feasible and a dual feasible with the same objective value, hence both are optimal

Back to the simplex algorithm

Recall Iteration 0 in our illustrative example

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

- Candidate dual solution: coefficients of the slack variables in row (0)
- Here: $y_1 = 0$, $y_2 = 0$, $y_3 = 0$
- Coefficient of the original primal variables x_1, x_2: indicate the slack in the dual constraints
  - Negative sign: dual constraints are violated
  - Indeed the solution $y_1 = 0$, $y_2 = 0$, $y_3 = 0$ violates all the constraints of the dual

Back to the simplex algorithm

Tableau at the end of Iteration 1

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	0	0	5/2	0	30
x_3	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
x_5	0	3	0	0	-1	1	6

- Candidate dual solution: $y_1 = 0$, $y_2 = 5/2$, $y_3 = 0$
- Coefficient of x_1 negative: indicates that the first dual constraint is violated
  - Indeed the current dual solution is infeasible, violating that $y_1 + 3y_3 \geq 3$

Back to the simplex algorithm

In general: look at row (0) in any iteration:

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	$z_1 - c_1$	$z_2 - c_2$	y_1	y_2	y_3	w

Interpretation:

- Initial iteration: coefficients of x_1 and x_2: $-c_1$ and $-c_2$ respectively
- z_1 and z_2: values added to the initial coefficients while running simplex
- But recall that c_1 and c_2 are also the right hand sides in the dual constraints
- $z_1 - c_1$: surplus variable for the first dual constraint
- What does simplex try to achieve? Nonnegative coefficients in all of row (0)
- In such a case: dual constraints satisfied, and dual variables nonnegative
- $\Rightarrow$ dual feasible solution with same value as primal feasible $\Rightarrow$ optimal solutions for both

Back to the simplex algorithm

Tableau at the end of Iteration 2

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	0	0	0	3/2	1	36
x_3	0	0	1	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

- Candidate dual solution: $y_1 = 0$, $y_2 = 3/2$, $y_3 = 1$
- All coefficients in row (0) nonnegative
- We can conclude that we have both a primal and a dual optimal solution
- Primal solution: $x_1 = 2$, $x_2 = 6$ read from right sides of last 2 rows

Back to the simplex algorithm

Advantages of using simplex for the dual?

- Suppose we have a LP with many constraints but few variables
- Dual of such an LP: many variables and few constraints
- We have seen that the complexity of simplex in practice seems to be proportional to the number of constraints
- Hence: it can be more beneficial in such cases to treat the dual as the linear program we want to solve

An Economic Interpretation of Dual Variables

Let us recall how we formulated our illustrative example

- A manufacturing company selling glass and aluminum products is trying to invest in launching 2 new products
- The company has 3 plants
  - Plant 1: for processing aluminum
  - Plant 2: for processing glass
  - Plant 3: for assembling and finalizing products
- Product 1 requires processing in Plant 1 and Plant 3
- Product 2 requires processing in Plant 2 and Plant 3
- Since the company processes other products as well, there are constraints on the amount of time available in each plant.

An Economic Interpretation of Dual Variables

As a result:

$$\max Z(x) = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 \le 18$$

$$x_1, x_2 \ge 0$$

- Variables: they express level of output for each product
- Coefficients in objective function: profit per unit of each product
- Right hand side parameters: the constraint for each available resource
- For this example: Resources $\Leftrightarrow$ Plants

An Economic Interpretation of Dual Variables

In general, consider a LP in standard form

$$\max Z(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

s.t.

$$A_i x \leq b_i, \text{ for } i = 1,\ldots, m$$

$$x_i \geq 0, \text{ for } i = 1,\ldots, n$$

Such problems typically arise by applications where:

- We have n products, m resources
- Variable x_j: expresses level of output of product j
- Coefficient c_j: profit per unit of product j
- Parameter a_{ij} from matrix A: how many units of resource i are needed per unit of product j
- Parameter b_i: Upper bound on the available amount of resource i

An Economic Interpretation of Dual Variables

In general, consider a LP in standard form

$$\max Z(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

s.t.

$$A_i\, x \le b_i, \text{ for } i = 1,\ldots, m$$

$$x_i \ge 0, \text{ for } i = 1,\ldots, n$$

Objective of the dual: $b_1y_1 + b_2y_2 + \ldots + b_my_m$

- Optimal dual solution has same value as the optimal profit
- Interpretation of dual variable y_i: contribution per unit of resource i to the total profit
- Hence, we can evaluate the effect on the profit by having b_i units of resource i available
- More importantly: we can estimate the change on the profit if we increase the availability of resource i by 1 unit

An Economic Interpretation of Dual Variables

- Back to our example:

$\max Z(x) = 3x_1 + 5x_2$

s.t.

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

$\min W(y) = 4y_1 + 12y_2 + 18y_3$

s.t.

$y_1 + 3y_3 \geq 3$

$2y_2 + 2y_3 \geq 5$

$y_1, y_2, y_3 \geq 0$

- Optimal dual solution: $y_1 = 0$, $y_2 = 3/2$, $y_3 = 1$
- Why is $y_1 = 0$?
- By complementary slackness, because the constraint $x_1 \leq 4$ is loose at the primal optimal ($x_1 = 2$)
- Even if we increase availability in Plant 1, we will not get a better solution!
  - Hence no need to consider changing the current usage of Plant 1

An Economic Interpretation of Dual Variables

Sensitivity analysis (or post-optimality analysis):

- Checking how solutions change as we vary the input parameters
- Very useful in operations research
  - Data may only represent estimates of the real parameters
  - We may also want to see if it is worth increasing the availability of some resources
- Do we need to solve the new LP from the beginning if we change e.g., the availability of a resource?
- It turns out we can save significantly in re-computing optimal solutions

An Economic Interpretation of Dual Variables

Sensitivity analysis (or post-optimality analysis):

Theorem:

- Consider a LP in the form

$\max \{ Z(x) = c^T \cdot x \mid A \cdot x \leq b, x \geq 0 \}$

- Let Z^* be the value of the optimal solution and $y_1, y_2, \ldots, y_m$ be an optimal dual solution
- Consider now a "perturbed" LP with each t_i "relatively small"

$$\max Z(x)$$

$$\text{s.t.}$$

$$A_i \cdot x \leq b_i + t_i, \text{ for } i = 1, \ldots, m$$

$$x \geq 0$$

- Then, new optimal = $Z^* + y_1 t_1 + y_2 t_2 + \ldots + y_m t_m$
- No need to re-solve the new LP

Further applications of Duality theory

Indicatively:

- Nonlinear programming: The duality framework can be generalized to convex programs or other forms of optimization problems

- Economic modeling and analysis
  - Computation of economic equilibria or pricing can be facilitated by the duality framework

- Design and analysis of algorithms, especially approximation algorithms for NP-hard problems
  - E.g., Primal-dual methods, LP-rounding methods
  - We will see some of these in later lectures

Further applications of Duality theory

Game theory: Computing Nash equilibria in zero-sum games

- One of the first applications of duality
- Initial proof for existence of equilibria by von Neumann did not yield an algorithm
- See Chapter 15 in [Hillier-Lieberman]