ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

M.Sc. Program in Data Science
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Optimization Techniques

Linear Programming – The Simplex Method

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The Simplex Method

- Designed by Dantzig (1947)
  - One of the most important algorithms of the 20th century
  - An algorithm that behaves extremely well in practice despite its exponential complexity in worst case
  - The design of the algorithm and the quest for better algorithms also contributed to building a rich theory around linear programming

Polyhedra

- Simplex is trying to optimize a linear function over a polyhedron
- Definition: In R^n, a polyhedron is defined by a set of linear inequalities on n variables

$$P = \{x: Ax \leq b\}$$

  – Where $x \in R^n$, $b \in R^m$, and A is an mxn matrix

- We will usually consider polyhedra in the form

$$P = \{x: Ax \leq b, x \geq 0\}$$

- A polyhedron is
  – Infeasible, if its feasible region is empty
  – Bounded, if there exists M, such that for every x in the feasible region, $||x||_2 \leq M$
  – Unbounded, if it is not bounded

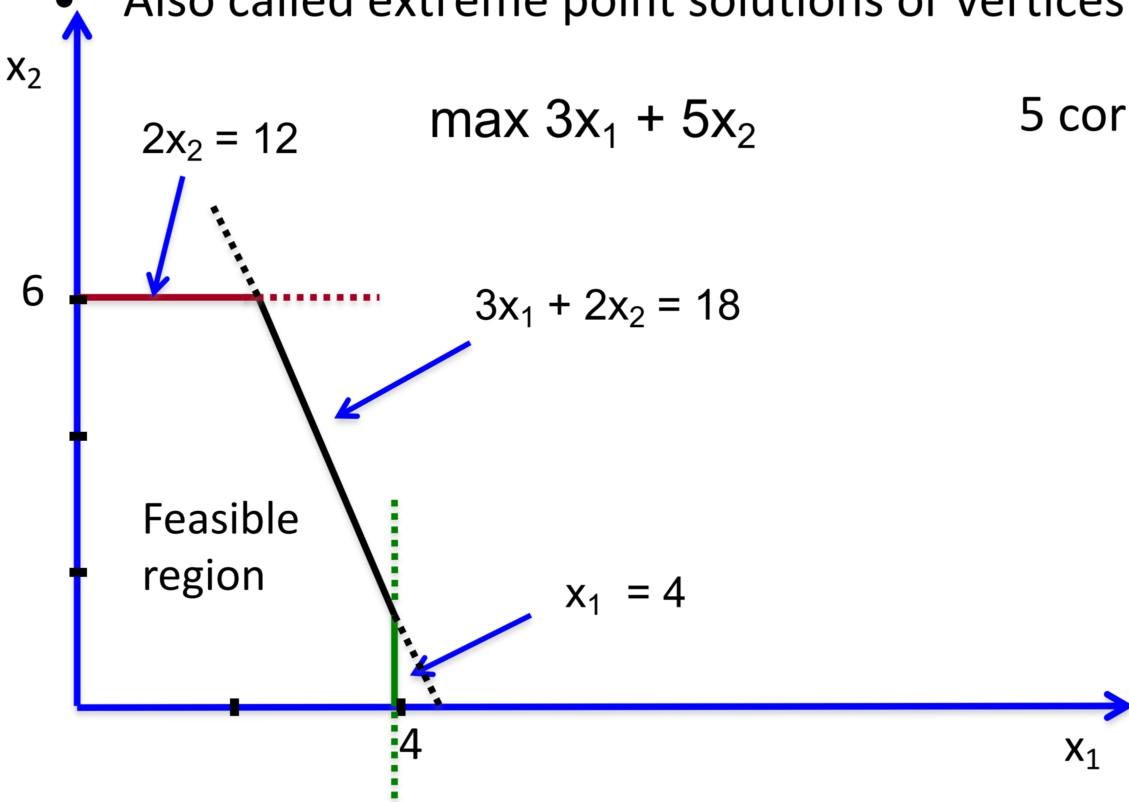
A Geometric Interpretation

- Simplex is an algebraic procedure
- However, it is important to understand its geometric motivation
- Assume the polyhedron is non-empty and bounded
  - Then, an optimal solution always exists for any linear objective function
  - A bounded polyhedron is also called polytope
- To illustrate the geometry of simplex, we will use Example 2 from Lecture 1 as a representative example in 2 dimensions

A Geometric Interpretation

Example 2: A polytope in R^2

- Constraint boundaries: correspond to the 5 sides of the polygon
- Corner point feasible (CPF) solutions: points at the intersection of constraint boundaries
- Also called extreme point solutions or vertices of the polytope



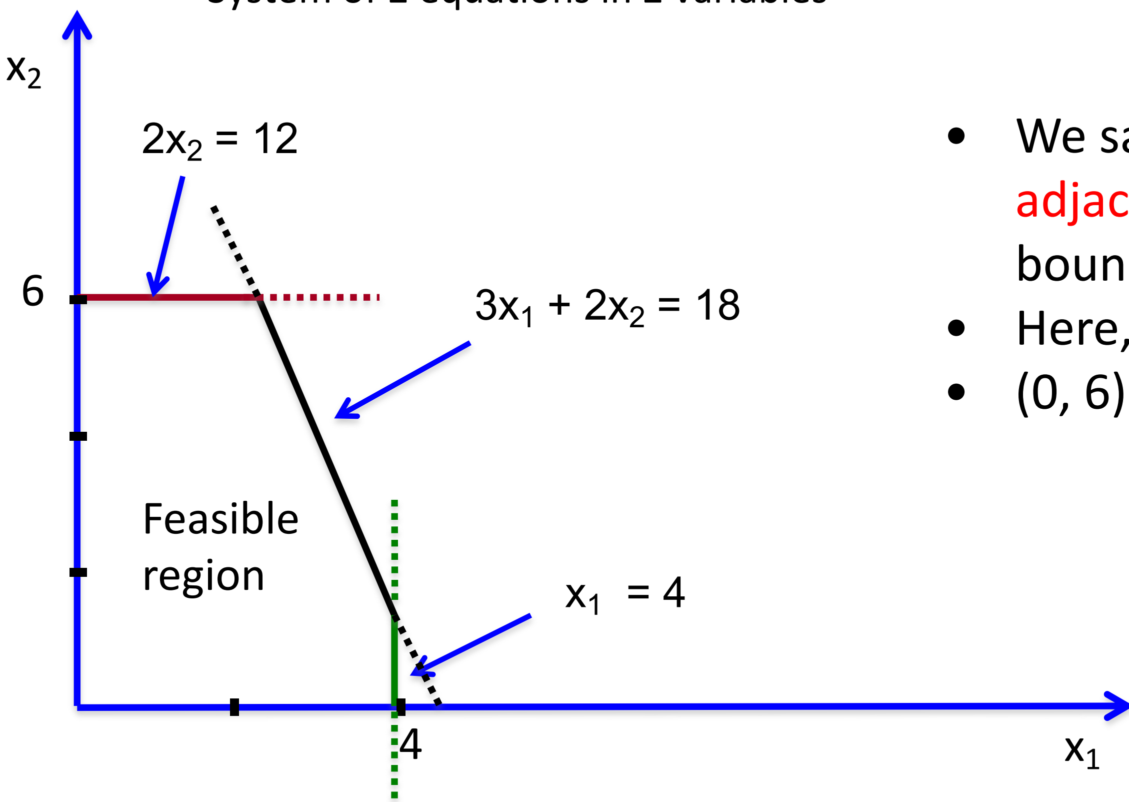
max $3x_1 + 5x_2$

5 corner point solutions here:
(0, 0),
(0, 6),
(2, 6),
(4, 3),
(4, 0)

A Geometric Interpretation

Example 2: A polytope in R^2

- Each corner point solution lies at the intersection of 2 constraint boundaries
- In 2 dimensions: how do we find each CPF solution?
  - System of 2 equations in 2 variables



- We say 2 corner point solutions are adjacent if they share 1 constraint boundary
- Here, (0, 0) and (0, 6) are adjacent,
- (0, 6) and (2, 6) are also adjacent

A Geometric Interpretation

Generalization to n dimensions:

In a polyhedron with n variables,

- a CPF solution is the intersection of n constraint boundaries
- How do we identify them?
  - system of n equations in n variables
  - Attention: make sure we have first removed "redundant" constraints
  - i.e., constraints that can be implied by linear combinations of the others (otherwise the system will not have a unique solution)
  - each group of n linearly independent constraints of the polyhedron yields a distinct CPF solution
- Two CPF solutions are adjacent if they share n-1 constraint boundaries

A Geometric Interpretation

Why are we interested in the notion of adjacent solutions?

Optimality test for linear programs:

Consider a LP with at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better, according to the objective function, then it must be an optimal solution.

- Hence, local optimality $\Rightarrow$ global optimality
- Extremely important property
  - Also generalizes to continuous, convex functions (to be discussed in next lectures)
- In our example: (2, 6) is an optimal solution
  - (2, 6) is adjacent to (0, 6) and (4, 3)
  - None of these achieve a better value for the objective function

A Geometric Interpretation

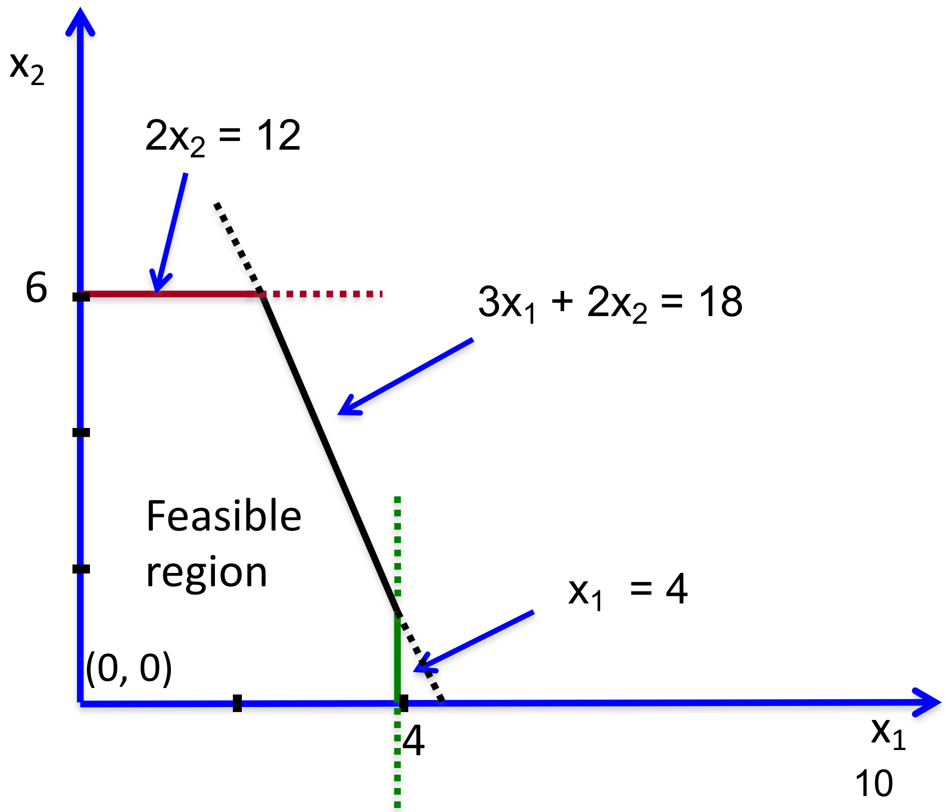
Outline of the simplex method from a geometric viewpoint

- Initialization: Choose an initial CPF solution
  - Usually we set all variables to 0
- Main iteration loop:
  - Apply the optimality test to the current CPF solution
  - If it is optimal stop,
  - else move to an adjacent solution that achieves the highest rate of increase in the objective function

A Geometric Interpretation

Solving Example 2 with the simplex method

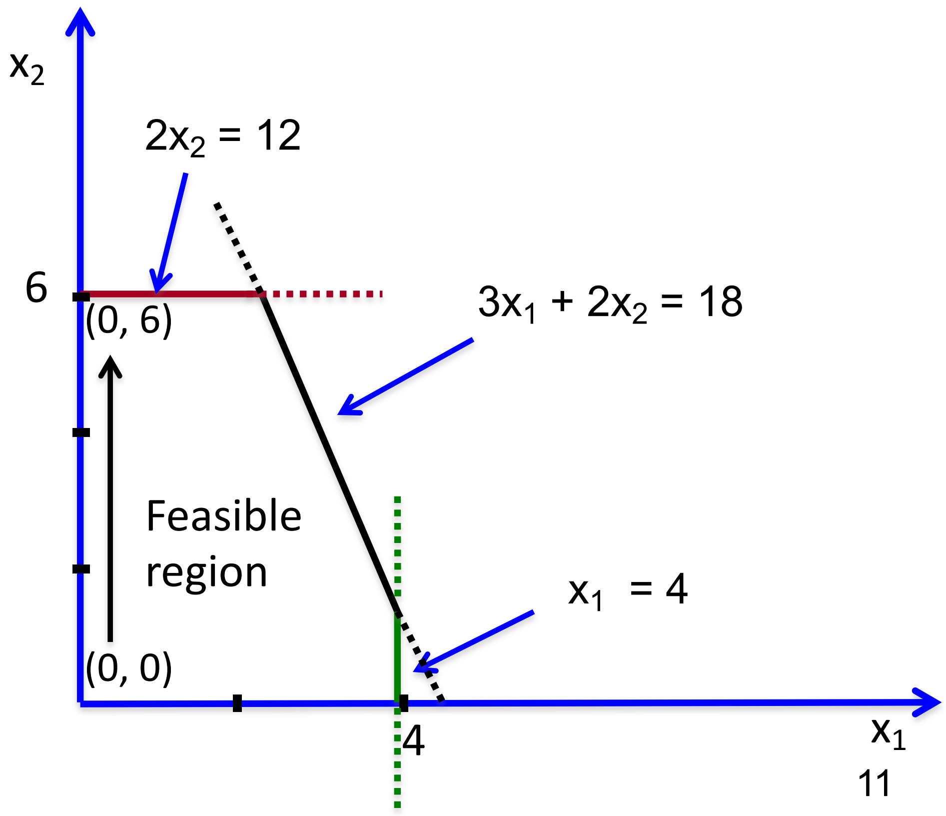
- Initialization:
  - we choose (0, 0) as the initial CPF solution
  - Optimality test: (0, 0) is not an optimal solution, there are better adjacent solutions



A Geometric Interpretation

Solving Example 2 with the simplex method

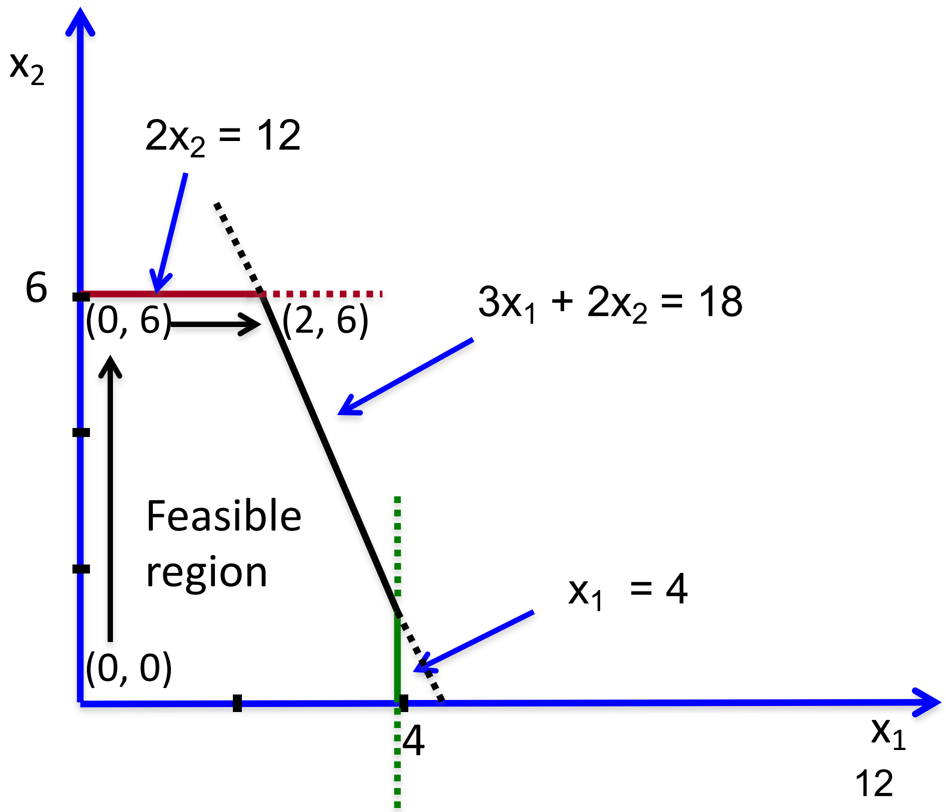
- Iteration 1:
  - Move from (0, 0) to an adjacent solution
  - How do we pick one?
  - Choose the direction that increases the objective function at a faster rate
  - Recall: $Z = 3x_1 + 5x_2$
  - Hence moving along the x_2 axis is better, stopping at (0, 6)
  - Optimality test: (0, 6) is not optimal



A Geometric Interpretation

Solving Example 2 with the simplex method

- Iteration 2:
  - Move from (0, 6) to a better adjacent solution
  - Moving back is not making things better
  - Hence, only choice to move to (2, 6)
  - Optimality test: (2, 6) is better than (0, 6) and (4, 3), therefore, it is an optimal solution



A Geometric Interpretation

Basic features of simplex

- It only examines CPF solutions
  - It is guaranteed that there always exists an optimal CPF solution
- Initialization: Whenever feasible, take (0, 0, ..., 0)
  - Nonnegativity constraints satisfied
  - What if the remaining constraints are violated? To be discussed again soon
- Picking the next CPF solution to visit:
  - Looking only at adjacent solutions can be easily implemented
  - The method only looks at the rate of increase in the objective function
  - Greedy local choice: we choose the direction with the best increase and stop at the adjacent solution in that direction

From Geometry to Algebra

Q: How can we implement all these steps in an automated algebraic manner for any polyhedron with n variables?

- We can use the geometric viewpoint only up to n=3 variables
- For n > 3, we need a translation into precise algebraic instructions

Setting up the simplex method

First step: Transform the standard form into a system of linear equations

- Conversion of inequality constraints into equalities by introducing *slack variables*
- For example: consider the inequality $x_1 \leq 4$ of Example 2
- We can define the slack variable: $x_3 = 4 - x_1$
- The constraint then is converted as:

$$x_1 \leq 4 \;\Rightarrow x_1 + x_3 = 4$$

- We can do this for all inequality constraints

Setting up the simplex method

Conversion of Example 2

- Need 3 slack variables: x_3, x_4, x_5

Original standard form

$$
\begin{aligned}
\text{max. } & Z = 3x_1 + 5x_2 \\
\text{s. t.:} & \\
& x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

$\Rightarrow$

Augmented form

$$
\begin{aligned}
\text{max. } & Z = 3x_1 + 5x_2 \\
\text{s. t.:} & \\
& x_1 + x_3 = 4 \\
& 2x_2 + x_4 = 12 \\
& 3x_1 + 2x_2 + x_5 = 18 \\
& x_i \geq 0,\ i = 1, \ldots, 5
\end{aligned}
$$

Algebraically, more convenient to work with the augmented form

Setting up the simplex method

Some terminology:

- Augmented solution: simply a solution for the original variables augmented by the slack variables
  - For the feasible solution (3, 2), the augmented solution is (3, 2, 1, 8, 5)
- Basic Feasible (BF) solution: an augmented CPF solution
  - (0, 6) is a CPF solution in the original problem
  - (0, 6, 4, 0, 6) is the corresponding BF solution
  - From a BF solution, we can get back the CPF solution by simply omitting the slack variables
- Understanding how BF solutions look like:
  - Example 2: 5 variables in total and 3 constraints
  - Hence, 2 degrees of freedom
  - If we set "arbitrary values" to 2 variables, then we can solve a linear system for the rest
  - In simplex: "arbitrary value" = 0

Setting up the simplex method

- For every BF solution:
  - We separate the variables into basic and nonbasic variables
  - Number of basic variables = m = number of constraints (excluding the nonnegativity constraints)
  - Number of nonbasic variables = n
  - Nonbasic variables are set to 0
  - Basic variables are then computed by solving the system of m linear equalities
  - The set of basic variables is referred to as the "basis" of the BF solution
- In our example:
  - (0, 0, 4, 12, 18) is a BF solution
  - Nonbasic variables: x_1, x_2, both set to 0
  - Basis = {x_3, x_4, x_5)
  - The values of the basis can be obtained by the constraints, after substituting $x_1 = x_2 = 0$

Setting up the simplex method

- Checking adjacency of two BF solutions
  - We could check if the corresponding CPF solutions are adjacent
  - Easier way: Two BF solutions are adjacent if their bases differ only in one variable (which means that all but one of their nonbasic variables are also the same)
- Illustration:
  - Adjacent CPF solutions: (0, 0) and (0, 6)
  - Corresponding BF solutions: S_1 = (0, 0, 4, 12, 18) and S_2 = (0, 6, 4, 0, 6)
  - In S_1, nonbasic variables = $\{x_1, x_2\}$, basis = $\{x_3, x_4, x_5\}$
  - In S_2, nonbasic variables = $\{x_1, x_4\}$, basis = $\{x_2, x_3, x_5\}$
  - Going from S_1 to S_2, variable x_2 switches from nonbasic to basic and variable x_4 leaves the basis
- Hence: very simple way of moving from one adjacent solution to another

Setting up the simplex method

Final step before running simplex:

- It becomes convenient to also treat the objective function as another equality constraint

$$\begin{aligned}
&\text{max. } Z \\
&\text{s. t.:} \\
&\quad Z - 3x_1 - 5x_2 = 0 \\
&\quad x_1 + x_3 = 4 \\
&\quad 2x_2 + x_4 = 12 \\
&\quad 3x_1 + 2x_2 + x_5 = 18 \\
&\quad x_i \geq 0, \quad i = 1, \ldots, 5
\end{aligned}$$

- No need for a slack variable since we have equality to begin with
- We will not really treat Z as a new variable

Algebraic description of the simplex method

Initialization:

- We need to choose an initial BF solution
  - In our example, setting $x_1 = 0$ and $x_2 = 0$ is feasible
  - Augmented solution: (0, 0, 4, 12, 18)
  - Hence, initial basis = $\{x_3, x_4, x_5\}$, nonbasic variables: x_1, x_2
- Optimality test:
  - Initial value of the objective function: Z = 0
  - Recall $Z = 3x_1 + 5x_2$, expressed as a function of the nonbasic variables
  - Coefficient for each nonbasic variable: rate of improvement for Z, if that variable were to be increased.
  - Here the rates of improvement are positive, which means the current BF solution is not optimal
  - Hence: simple way to answer the optimality test

Algebraic description of the simplex method

Iteration 1:

- We need to determine the direction of movement towards an adjacent BF solution
  - Coefficient of x_2 in Z > coefficient of x_1
  - We pick x_2 as the variable to increase
  - Variable x_2 will enter the basis (referred to as the entering basic variable)
- How much shall we increase x_2 ?
  - For as long as we do not violate the constraints!

(1) $x_1 + x_3 = 4 \Rightarrow x_3 = 4$

(2) $2x_2 + x_4 = 12 \Rightarrow x_4 = 12 - 2x_2$

(3) $3x_1 + 2x_2 + x_5 = 18 \Rightarrow x_5 = 18 - 2x_2$

And now use the nonnegativity constraints!

Algebraic description of the simplex method

Iteration 1:

(1) $x_1 + x_3 = 4 \quad \Rightarrow x_3 = 4$

(2) $2x_2 + x_4 = 12 \quad \Rightarrow x_4 = 12 - 2x_2$

(3) $3x_1 + 2x_2 + x_5 = 18 \quad \Rightarrow x_5 = 18 - 2x_2$

$x_3 \geq 0 \quad \Rightarrow$ no upper bound on x_2

$x_4 \geq 0 \quad \Rightarrow 12 - 2x_2 \geq 0 \Rightarrow x_2 \leq 6$

$x_5 \geq 0 \quad \Rightarrow 18 - 2x_2 \geq 0 \Rightarrow x_2 \leq 9$

- We pick the minimum value implied by the upper bounds
- Increasing x_2 beyond the value of 6 would result in an infeasible solution
- Hence, we stop at $x_2 = 6$

Algebraic description of the simplex method

Iteration 1:

- Can we arrive at $x_2 = 6$ with a more automated way?
- Minimum Ratio Test:
  - For each constraint, divide the constant term by the coefficient of x_2
  - The minimum such ratio tells us how much to increase x_2

(1) $x_1 + x_3 = 4 \Rightarrow$ ratio = 4/0 = $+\infty$ (0 coefficient of x_2)

(2) $2x_2 + x_4 = 12 \Rightarrow$ ratio = 12/2 = 6

(3) $3x_1 + 2x_2 + x_5 = 18 \Rightarrow$ ratio = 18/2 = 9

- Setting $x_2 = 6$ makes variable x_4 drop to 0
- x_4 is called the *leaving basic variable*

Algebraic description of the simplex method

Iteration 1:

- Summarize what we have done so far:

	Initial BF solution	New BF solution
Nonbasic variables	$x_1 = 0, x_2 = 0$	$x_1 = 0, x_4 = 0$
Basis	$x_3 = 4, x_4 = 12, x_5 = 18$	$x_2 = 6, x_3 = ?, x_5 = ?$

- Final step of Iteration 1:
  - Convert the system of equations according to the new basis
  - Express the objective function in terms of the new nonbasic variables
  - Compute the missing values in the new BF solution (for x_3 and x_5)

Algebraic description of the simplex method

Iteration 1:

Initial constraints

(0) $Z - 3x_1 - 5x_2 = 0$

(1) $x_1 + x_3 = 4$

(2) $2x_2 + x_4 = 12$

(3) $3x_1 + 2x_2 + x_5 = 18$

- We need x_2 to disappear from (0), (1) and (3)
- Start with row (2): row (2) /2 $\Rightarrow$ $x_2 + 1/2\, x_4 = 6$
- We can then
  - multiply a row by a constant
  - Add/subtract multiples of a row to/from another row
  - For example: row (0) := row (0) + 5 · row (2)

Algebraic description of the simplex method

Iteration 1:

Final set of constraints at the end of the iteration

(0) $Z - 3x_1 + 5/2\ x_4 = 30$

(1) $x_1 + x_3 = 4$

(2) $x_2 + 1/2\ x_4 = 6$

(3) $3x_1 - x_4 + x_5 = 6$

- Procedure for obtaining the new form of the constraints: the Gauss-Jordan method
- Hence, assignment of values in the new BF solution:
  - $x_1 = 0$, $x_4 = 0$, $x_2 = 6$, $x_3 = 4$, $x_5 = 6$
- Optimality test:
  - $Z = 30 + 3x_1 - 5/2x_4$, positive coefficient for $x_1 \Rightarrow$ not optimal
  - Hence, we need to move to an adjacent BF solution

Algebraic description of the simplex method

Iteration 2:

- Which variable should now enter the basis?
  - Unique choice: Coefficient of x_1 is the only positive coefficient in Z
  - Variable x_1 is now the new entering basic variable
- How much shall we increase x_1 ?
  - Apply the Minimum Ratio Test
  - Set $x_1 := 2$ due to equation (3)
- Which variable exits the basis?
  - Again, from the Minimum Ratio Test, x_5 will be the leaving variable
- New basis: $\{x_1, x_2, x_3\}$
  - Nonbasic variables: $x_4 = x_5 = 0$
  - New BF solution: (2, 6, 2, 0, 0)

Algebraic description of the simplex method

Iteration 2:

Substituting using the Gauss-Jordan method:

(0) $Z + 3/2\ x_4 + x_5 = 36$

(1) $x_3 + 1/3\ x_4 - 1/3\ x_5 = 2$

(2) $x_2 + ½\ x_4 = 6$

(3) $x_1 - 1/3\ x_4 + 1/3\ x_5 = 2$

- Optimality test:
  - $Z = 36 - 3/2\ x_4 - x_5$
  - There is no direction of improvement, increasing x_4 or x_5 will decrease the objective function
  - Current BF solution is optimal
  - Solution of the original linear program: $x_1 = 2$, $x_2 = 6$ and $Z = 36$

Tabular form of the simplex method

- So far we have managed to transform our geometric intuition into an algebraic procedure
- Operations used pretty simple
- Nevertheless, we can make the process even more automatizable
- Simplex tableau: A tabular representation of the constraints and the current BF solution
- All we need to know: the basis and the coefficients in each row

Tabular form of the simplex method

Algebraic form vs tableau:

Let us revisit the initialization in our Example:

(0) $Z - 3x_1 - 5x_2 = 0$

(1) $x_1 + x_3 = 4$

(2) $2x_2 + x_4 = 12$

(3) $3x_1 + 2x_2 + x_5 = 18$

← Algebraic form

Corresponding tableau form

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

rows = number of constraints + 1

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

- For notational convenience: treat Z also as a basic variable
- Optimality test in a tableau:
  - We have reached an optimal solution when the coefficients in row (0) are all nonnegative

Tabular form of the simplex method

Basis	Coefficients						Right side	
	Z	x_1	x_2	x_3	x_4	x_5		
Z	1	-3	-5	0	0	0	0	
x_3	0	1	0	1	0	0	4	
x_4	0	0	2	0	1	0	12	12/2 = 6
x_5	0	3	2	0	0	1	18	18/2 = 9

Iteration 1:

- Which variable should enter the basis?
  - The nonbasic variable with the most negative coefficient in row (0), hence x_2
  - Column of x_2: **pivot column**
- Minimum Ratio Test
  - How do we run it?
  - Information we need is the right side column and the column of x_2

Tabular form of the simplex method

Basis	Coefficients						Right side	
	Z	x_1	x_2	x_3	x_4	x_5		
Z	1	-3	-5	0	0	0	0	
x_3	0	1	0	1	0	0	4	
x_4	0	0	2	0	1	0	12	12/2 = 6
x_5	0	3	2	0	0	1	18	18/2 = 9

Iteration 1:

- Outcome of the Minimum Ratio Test
  - Minimum achieved at row of x_4
  - Leaving variable: x_4, i.e., the basic variable corresponding to that row
  - Row of x_4: the **pivot row**
  - Intersection of pivot row and pivot column: **pivot element** (=2 in this iteration)

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	-5	0	0	0	0
x_3	0	1	0	1	0	0	4
x_4	0	0	2	0	1	0	12
x_5	0	3	2	0	0	1	18

12/2 = 6

18/2 = 9

Iteration 1:

- Final step: Gauss-Jordan method to get the new tableau
  - Divide first the pivot row by the pivot element
  - This makes the coefficient of x_2 equal to 1 in the pivot row
  - Then we can add/subtract appropriate multiples of the pivot row to the other rows (just as in the algebraic description of simplex)

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	-3	0	0	5/2	0	30
x_3	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
x_5	0	3	0	0	-1	1	6

New tableau

End of Iteration 1:

- Optimality test:
  - There exists a negative coefficient in row (0)
  - Hence, we need to go to the next iteration

Tabular form of the simplex method

Basis	Coefficients						Right side	
	Z	x_1	x_2	x_3	x_4	x_5		
Z	1	-3	0	0	5/2	0	30	
x_3	0	1	0	1	0	0	4	4/1 = 4
x_2	0	0	1	0	1/2	0	6	
x_5	0	3	0	0	-1	1	6	6/3 = 2

Iteration 2:

- Which variable should enter the basis?
  - Only variable x_1 has a negative coefficient in row (0)
  - **Pivot column:** The column of x_1
- Minimum Ratio Test
  - Variable x_5 is the leaving variable
  - **Pivot row:** The row of x_5

Tabular form of the simplex method

Basis	Coefficients						Right side
	Z	x_1	x_2	x_3	x_4	x_5	
Z	1	0	0	0	3/2	1	36
x_3	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

New tableau

End of Iteration 2:

- Optimality test
  - No negative coefficient in the row of Z
  - Hence we stop at the current BF solution (2, 6, 2, 0, 0)
  - Optimal solution to the original problem: x_1 = 2, x_2 = 6

Summary: Geometric, algebraic and tableau form

We have seen 3 different ways of thinking about the same algorithm

- **Geometric view:**
  - This is how the algorithm was inspired
  - Useful only for 2 or 3 dimensions
- **Algebraic description**
  - More convenient for learning the logic of the algorithm esp. in higher dimensions
- **Tableau form**
  - Equivalent to the algebraic form in terms of operations performed
  - However, it organizes the data in a more compact form
  - Allows for better automatization

Tie-breaking and other technical details

Some issues that may arise

- **During the execution:**
  - Many choices for the entering variable
  - Many choices for the leaving variable
  - No leaving variable
- **At initialization:**
  - Difficulty in finding an initial feasible solution to begin with
- **At termination:**
  - Multiple optimal solutions

Tie-breaking and other technical details

- **Many choices for the entering variable:**
  - No problem, make an arbitrary choice
  - An optimal solution will be reached eventually
  - Hard to know in advance which one is the best choice

Tie-breaking and other technical details

- **Many choices for the leaving variable:**
  - –This may cause problems
  - –All such variables will become 0 at the end of the iteration
  - –Hence, we will have some basic variables with a 0 value
  - –Such solutions are called *degenerate*
  - –They may not allow Z to increase in the next iteration
  - –The algorithm may get trapped in a loop where some variables enter and exit the basis repeatedly and Z gets stuck at the same value
  - –Bland's rule: If there are multiple candidate leaving variables, always choose the variable with the smallest index
  - –Also: rarely been observed in practice, almost safe to ignore this

Tie-breaking and other technical details

- **No leaving basic variable:**
  - This means that the entering variable can be increased indefinitely
  - The increase does not yield any negative values to the current basic variables
  - In the tableau form: all coefficients in pivot column are negative or 0 (except first row)
  - Conclusion: The problem is unbounded, optimal solution is $+\infty$
  - Maybe a mistake has occured in the initial formulation of the problem

Tie-breaking and other technical details

- **Multiple optimal solutions:**
  - If there are multiple optimal solutions, there are at least 2 optimal CPF solutions
  - Any convex combination of these CPF solutions is also an optimal solution
  - In some problems we may only care to identify one optimal solution and stop
  - If we care to find all optimal CPF solutions:
    - Run more iterations of simplex after we found the first optimal solution
    - Choose a nonbasic variable with zero coefficient in the row of Z as the entering variable
    - There exists such a variable whenever there are multiple optimal solutions

Tie-breaking and other technical details

- **Difficulty in finding an initial feasible solution:**
  - What if the all-0 solution is not feasible? How do we start simplex then?
  - This can happen when some coefficients b_i are negative
  - Strategy: Define an auxiliary problem so that
    - It is easy to find an initial feasible solution in the auxiliary problem
    - The optimal solution of the auxiliary can tell us whether there exists a feasible solution in our original problem
  - 2-phase simplex method:
    - First run simplex on the auxiliary problem
    - See whether we can identify an initial basic feasible solution from the optimal solution of the auxiliary problem
    - If yes, run simplex on our original problem

Tie-breaking and other technical details

- **Difficulty in finding an initial feasible solution:**
  - There are various ways to define the auxiliary problem
  - Illustration:

$$\begin{aligned} &\max Z = c^T x \\ &\text{s.t.} \\ &\quad \Sigma_j a_{ij}x_j \le b_i \\ &\quad x_i \ge 0, i = 1,\ldots,n \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min 1^T y \\ &\text{s.t.} \\ &\quad \Sigma_j a_{ij}x_j + y_i = b_i , i = 1,\ldots n \\ &\quad x_i \ge 0, y_i \ge 0, i = 1, \ldots,n \end{aligned}$$

- The auxiliary problem always has a feasible solution
  - Set original variables to 0, and y equal to b.
- The original problem has a feasible solution if and only if the optimal of the auxiliary is 0

Other variants in implementing Simplex

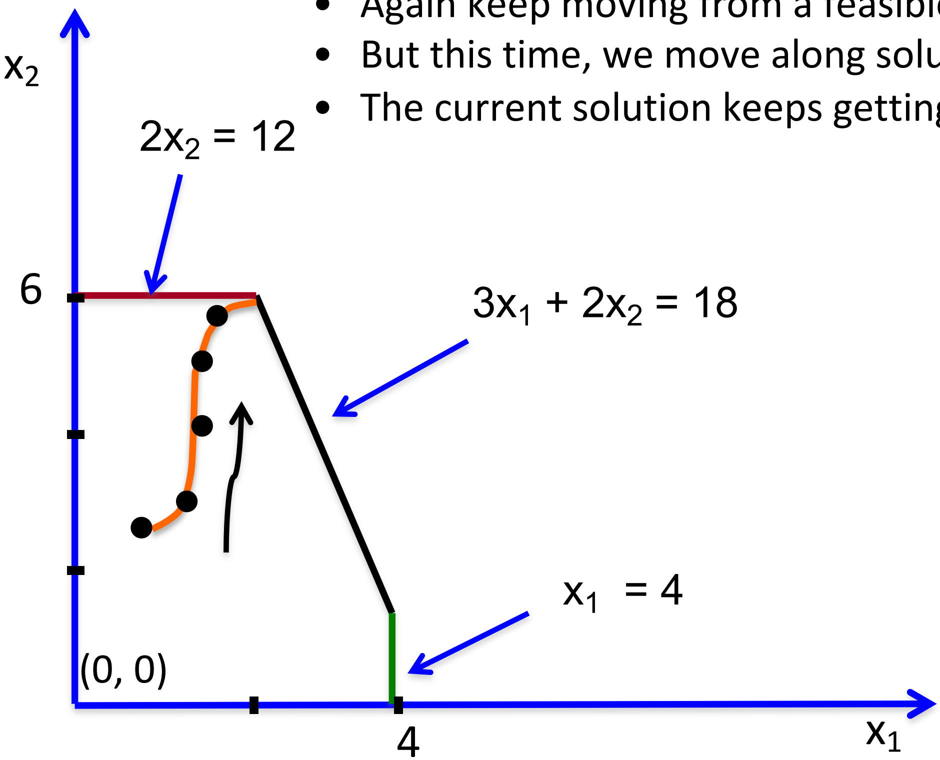
- The revised simplex method
  - Based on exploiting fast matrix operations
  - Each iteration requires solving 2 systems of linear equations
  - But these are not solved from scratch
  - Only small updates based on the solution from previous iteration
- The dual simplex method
  - Applying simplex to the dual linear program
  - But essentially working with the primal
  - Useful tool for sensitivity analysis
- Many other variations have also been suggested over the years...

Complexity of Simplex

- Extremely well-behaved in practice
- Empirically, number of iterations in simplex looks proportional to number of constraints, e.g., usually no more than 3m
- Can we have a good theoretical upper bound on the number of iterations?
- NO! There are examples that need an exponential (2^n) number of iterations, discovered first by [Klee, Minty '72]
- Despite that, it is still one of the preferred algorithms for solving linear programs!

Other Algorithms

- **The ellipsoid method:** The first polynomial time algorithm
  - By [Kachiyan '79], however not well behaved in practice
- **Interior point methods:** also polynomial time algorithms
  - First conceived by Karmarkar [1984]
  - Main ideas:
    - Again keep moving from a feasible solution to a better one
    - But this time, we move along solutions in the interior of the polytope
    - The current solution keeps getting closer and closer to a vertex of the polytope



Other Algorithms

Types of interior point methods

- Affine scaling algorithms
  - One of the simplest interior point algorithms
  - Based on approximating polyhedra by “ellipsoids”
  - Optimizing over ellipsoids in each iteration
  - Non-linear problems but solvable with closed form solutions
- Potential reduction algorithms
  - Do not measure progress by the increase in the objective function
  - Instead use a non-linear potential function
- Path following algorithms
  - Transforms the initial problem into an unconstrained problem (or a problem with equality constraints)
  - Incorporates the inequality constraints “$x_i \geq 0$” into the objective function (logarithmic barrier function)
  - Solves the resulting non-linear problem with Newton’s method

Simplex vs Interior Point Algorithms

- **Comparisons**
  - In theory: interior point methods are polynomial time algorithms (for any n and m), simplex may need exponential time
  - In practice: average case complexity of simplex very low compared to worst case
  - One iteration of interior point methods needs much more computation time than in simplex to decide the next feasible solution
  - But: as the number of constraints increases, interior point methods do not need much more iterations
  - Number of iterations in simplex may increase rapidly as we increase the number of variables and constraints
    - Interior point methods go through the internal part of the polytope
    - Adding more constraints reduces the feasible region, by adding more constraint boundaries
    - Hence, for problems with many thousands of constraints, interior point methods seem to be the best hope