ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

M.Sc. Program in Data Science
Department of Informatics

Optimization Techniques

Linear Programming

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Introduction to Linear Programming

Linear Programming

What we will see:

- Formulating optimization problems as linear programs

- Algorithms for solving linear programs
  - Graphical method
  - Simplex and its variants
  - Short discussion on other methods (interior point methods)

- Duality theory

Linear Programming

- Nothing to do with programming!
- A particular way of formulating certain optimization problems with linear constraints and a linear objective function
- One of the most useful tools in Algorithms and Operations Research
- Extremely useful also in the design of mathematical models for combinatorial optimisation problems as well as LP-based heuristics

Linear Programming

Applications of Linear Programming: Too many to enumerate!

- Operations Research
- Theory of Algorithms and Combinatorial Optimization
- Game theory and Microeconomics
- Manufacturing
- Logistics
- Medicine
- And many more...

Linear Programming Examples

Example 1:

- A farmer possesses a land of 10 km^2
- He wants to plant the land with wheat, or barley or a combination of them
- The farmer has a limited amount of fertilizer, say 16 kgs
- And a limited amount of pesticide, say 18 kgs
- Each square km of wheat requires 1kg of fertilizer and 2 kgs of pesticide
- Each square km of barley requires 2kg of fertilizer and 1.2 kgs of pesticide
- Revenue to the farmer: 3 (thousand €) from each square km of wheat and 4 (thousand €) from each square km of barley
- Find out what the farmer should do (i.e., how many square km of barley and how many of wheat he should plant)

Linear Programming Examples

Formulation as a linear program:

First step: We need to define the decision variables of our problem

- x_1 = number of square km for wheat
- x_2 = number of square km for barley
- Often multiple ways for doing this step
- Objective function: maximize $3x_1 + 4x_2$
- Observe that: objective function is linear

Linear Programming Examples

Formulation as a linear program:

Second step: formulation of constraints on the variables x_1, x_2

- Area constraint: $x_1 + x_2 \leq 10$
- Constraint for fertilizer: $x_1 + 2x_2 \leq 16$
- Constraint for pesticide: $2x_1 + 1.2x_2 \leq 18$
- Nonnegativity constraints: $x_1 \geq 0$, $x_2 \geq 0$ (cannot plant an area with negative surface)
- Observe: all constraints are also linear

Linear Programming Examples

Usual writing style:

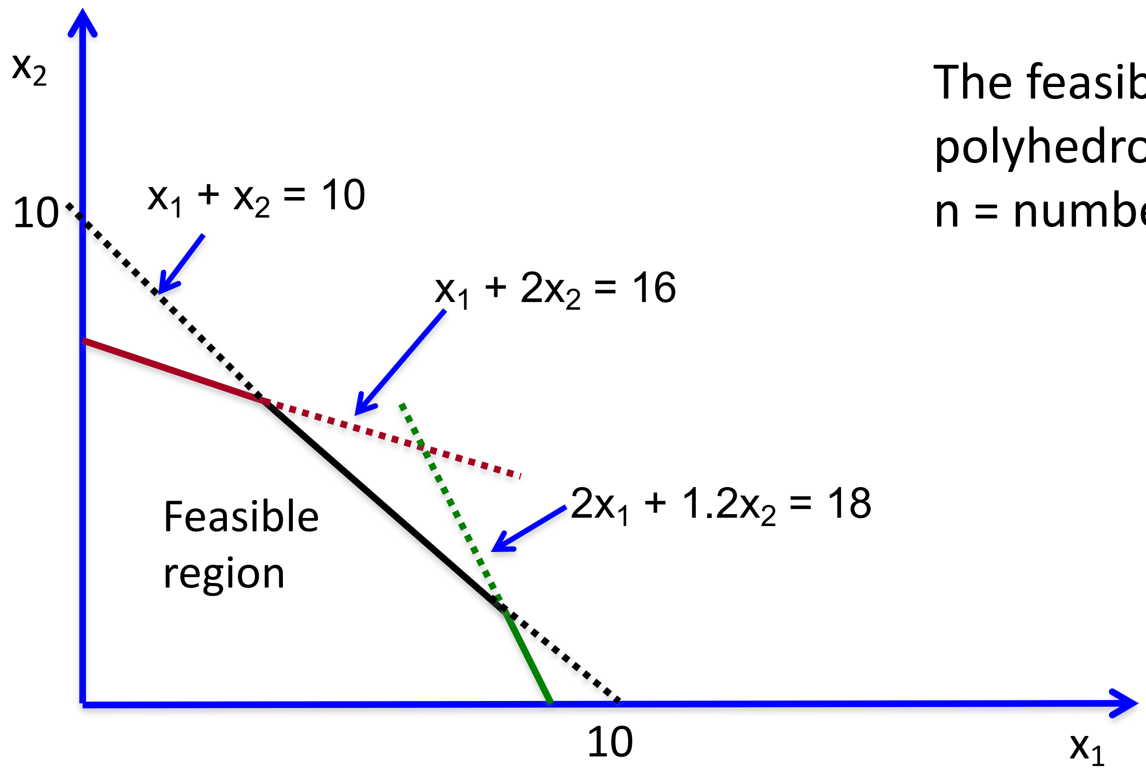
$$\max\ 3x_1 + 4x_2 \quad \longleftarrow \text{Objective function}$$

$$\left.\begin{aligned} s.t.\quad & x_1 + x_2 \le 10 \\ & x_1 + 2x_2 \le 16 \\ & 2x_1 + 1.2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned}\right\} \text{constraints}$$

- It can be either a minimization or a maximization problem
- Feasible space (or region): the set of all pairs (x_1, x_2) that satisfy the constraints
- In the example: the feasible region is a subset of R^2

Linear Programming Examples

Geometrically:



The feasible region is a polyhedron in R^n, where n = number of variables

Linear Programming Examples

Example 2:

- A manufacturing company selling glass and aluminum products is trying to invest in launching 2 new products
- The company has 3 plants
  - Plant 1: for processing aluminum
  - Plant 2: for processing glass
  - Plant 3: for assembling and finalizing products
- Product 1 requires processing in Plant 1 and Plant 3
- Product 2 requires processing in Plant 2 and Plant 3
- Since the company processes other products as well, there are constraints on the amount of time available in each plant.

Linear Programming Examples

Plant	Time needed per batch (hours)		Total available time per week (hours)
	Product		
	1	2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	3000	5000	

- Goal: Decide how many batches of Product 1 and Product 2 to produce so as
  - Not to exceed the available time capacity in each plant
  - Maximize total revenue from the batches produced

Linear Programming Examples

Formulation as a linear program:

First step: determine the decision variables of our problem

- x_1 = number of batches of product 1, produced per week
- x_2 = number of batches of product 2, produced per week

Second step: formulation of constraints on the variables x_1, x_2

- Time constraints for Plant 1: $x_1 \leq 4$
- Time constraints for Plant 2: $2x_2 \leq 12$
- Time constraints for Plant 3: $3x_1 + 2x_2 \leq 18$
- Nonnegativity constraints: $x_1 \geq 0$, $x_2 \geq 0$ (number of batches produced cannot be negative)

Objective function: maximize $3x_1 + 5x_2$

Linear Programming Examples

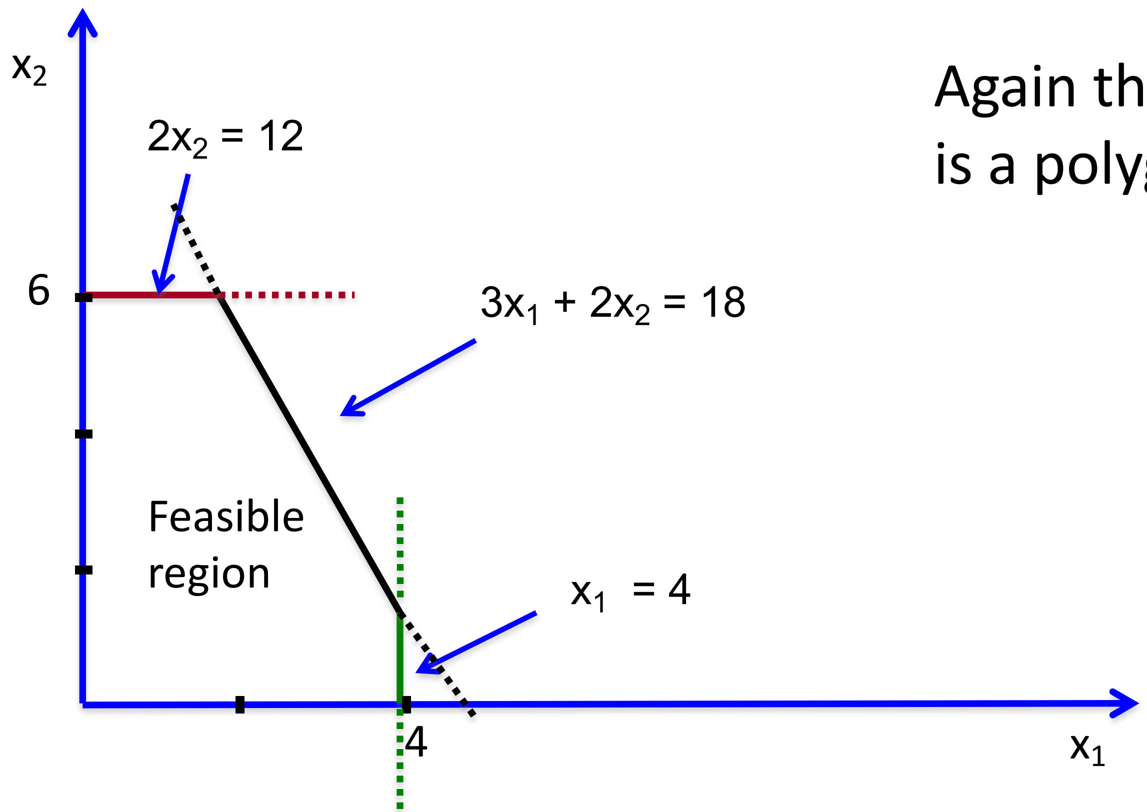
Hence:

$$\max \; 3x_1 + 5x_2 \quad \longleftarrow \quad \text{Objective function}$$

$$\left.\begin{aligned} s.t. \quad & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned}\right\} \quad \text{constraints}$$

Linear Programming Examples

Geometrically:



Again the feasible region is a polygon in R^2

Linear Programming Examples

A more succinct notation

We can represent Example 2 as:

$$\begin{aligned} \text{max.} \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$where\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ c = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix},\ A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix}$$

Notation: $x \geq 0$ for a vector x means that the inequality should hold component-wise (for every coordinate)

Linear Programming Examples

Example 3:

- A team of doctors are trying to design a radiation therapy for a certain group of cancer patients.
- The doctors will use 2 beams
- The dose of radiation should be high enough to kill tumor cells but should also not kill too many healthy cells or cause any other problems
- Constraints arise from the amount of radiation absorbed in the area around the tumor, which may involve
  - Healthy anatomy
  - Critical tissues
  - Tumor region
- Real application in prostate cancer:
  - Collaboration between Memorial Sloan-Kettering Cancer Center and Center for Operations Research in Medicine and Health Care
  - 2007 first award for “Achievement in Operations Research and Management Sciences”

Linear Programming Examples

Area	Average fraction of dose absorbed (kilorads)		Restriction on total average dosage (kilorads)
	Beam 1	Beam 2	
Healthy anatomy	0.4	0.5	
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	= 6
Center of tumor	0.6	0.4	≥ 6

- Goal: Minimize the total dosage reaching the healthy anatomy

Linear Programming Examples

Hence:

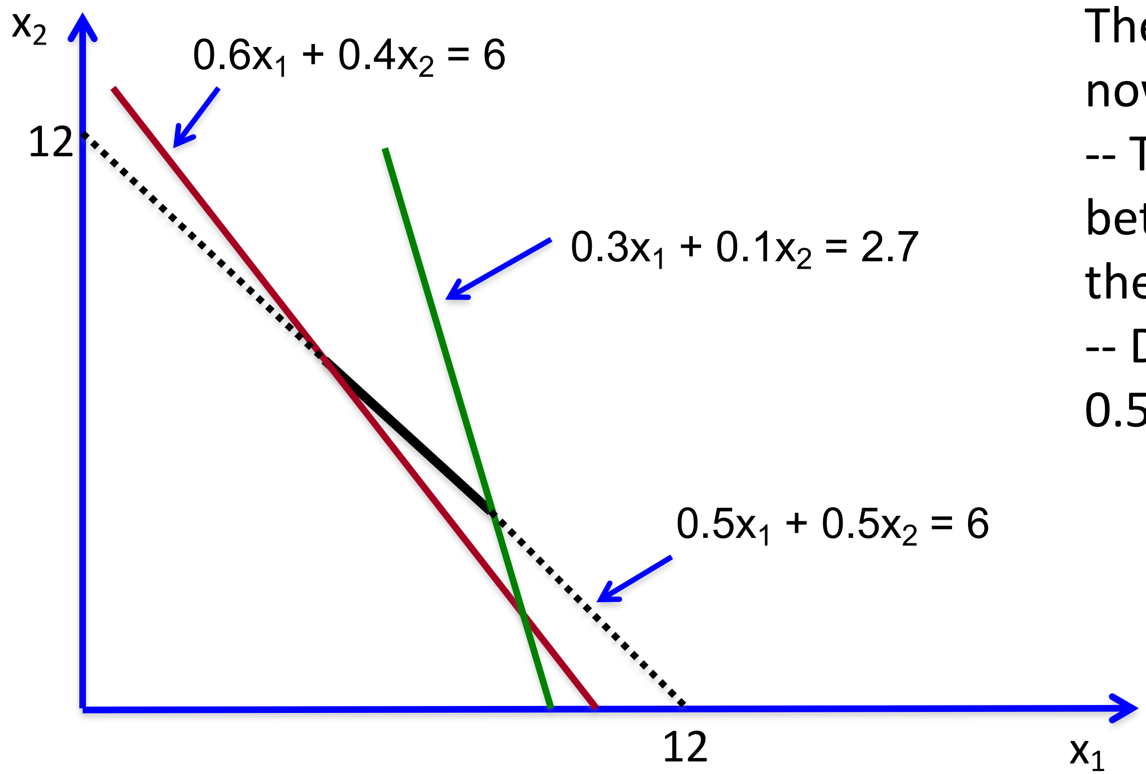
$$
\begin{aligned}
\min\ & 0.4x_1 + 0.5x_2 \\
s.t.\ & 0.3x_1 + 0.1x_2 \le 2.7 \\
& 0.5x_1 + 0.5x_2 = 6 \\
& 0.6x_1 + 0.4x_2 \ge 6 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Minimization instead of maximization

Different types of constraints from previous examples

Linear Programming Examples

Geometrically:



The feasible region is now a line
-- The black segment between the red and the green line
-- Due to the constraint $0.5x_1 + 0.5x_1 = 6$

Our Standard Form of Linear Programs

From now on, we will focus on problems in the format of Examples 1 and 2

Given:

- $c_1, c_2, \ldots, c_n$
- $b_1, b_2, \ldots, b_m$
- The constraint matrix $A = (a_{ij})$ with $1 \leq i \leq n$, $1 \leq j \leq m$,

We want to:

$$
\begin{aligned}
&\text{maximize } Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
&\text{subject to:} \\
&\qquad a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
&\qquad a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
&\qquad \vdots \\
&\qquad a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m \\
&\qquad x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0
\end{aligned}
$$

Our Standard Form of Linear Programs

More concisely:

$$\begin{aligned} \text{max:}\ & Z = c^T \cdot x \\ \text{s. t.:}\ & \\ & A \cdot x \leq b \\ & x \geq 0 \end{aligned}$$

Where:

- c and x are n-dimensional vectors
- b is an m-dimensional vector
- n decision variables
- m inequality constraints
- n nonnegativity constraints

Linear Programming

Other forms of LPs we could encounter:

1. Minimization problem instead of maximization
2. >= inequalities in the constraints
3. Equality constraints
4. Absence of nonnegativity constraints

Claim: All these are equivalent forms, and can be brought to our standard form

- If we have a minimization problem: revert the signs in the coefficients of the objective function and maximize the new function.
- >= constraints: again revert signs
- Equality constraints: replace them by 2 constraints (one with >=, and one with <=)

The Graphical Method

- Applicable for linear programs with 2 decision variables
- It helps us understand how to think about solving problems in higher dimensions

Solving Example 2:

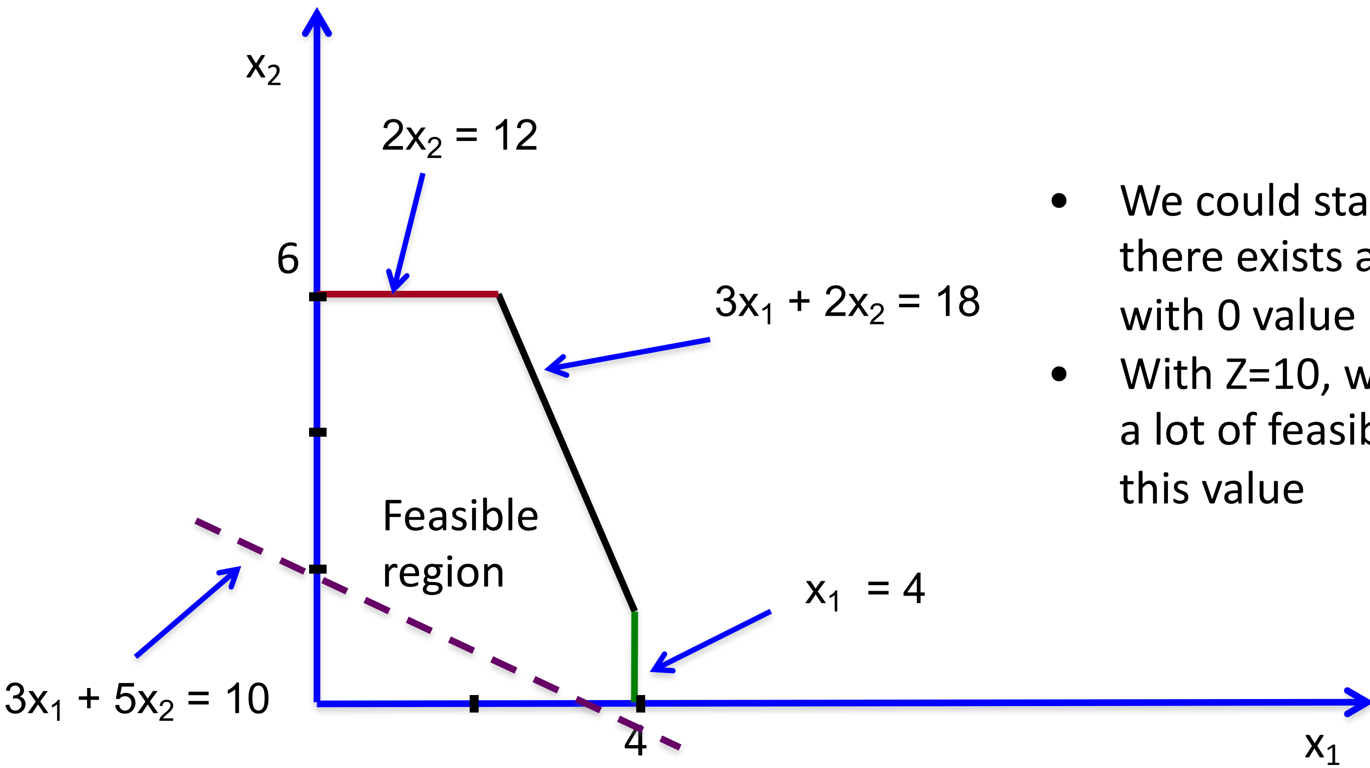
- Step 1: Draw the feasible region
- Step 2: "Guess" a value Z for the objective function and draw the line $3x_1 + 5x_2 = Z$
- If this line intersects the feasible region, it means we have at least one feasible solution with value Z
- Trial and error: Keep doing this, increasing Z till the line gets out of the feasible region

The Graphical Method

Solving Example 2:

•Step 1: Draw the feasible region

•Step 2: Trial and error: "Guess" a value Z for the objective function and draw the line $3x_1 + 5x_2 = Z$

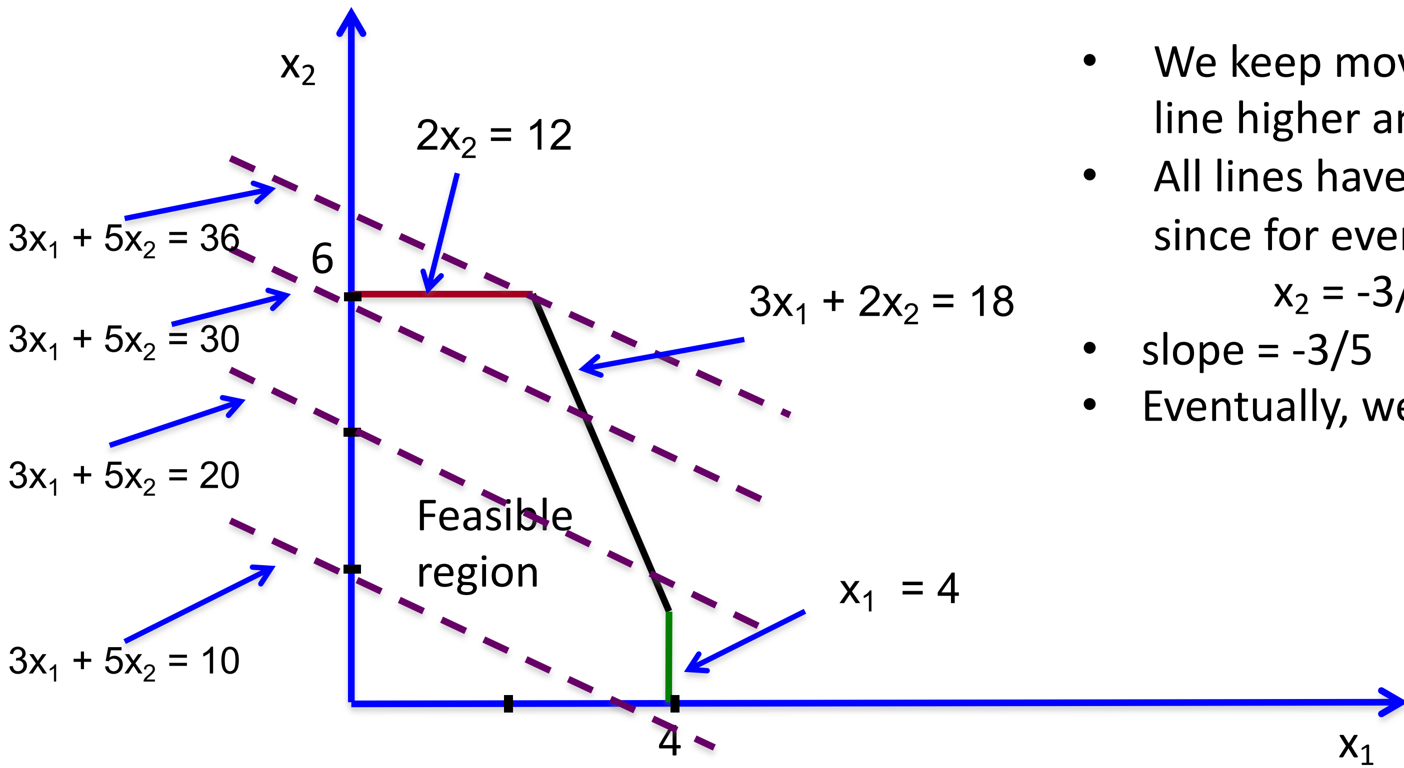


- We could start with Z=0 since there exists a feasible solution with 0 value
- With Z=10, we see there are still a lot of feasible solutions with this value

The Graphical Method

Solving Example 2:

- We can now keep examining higher values for Z, until we get out of the feasible region



- We keep moving the dashed line higher and higher
- All lines have the same slope, since for every Z:

  $x_2 = -3/5\ x_1 + 1/5\ Z$
- slope = -3/5
- Eventually, we stop at Z = 36

The Graphical Method

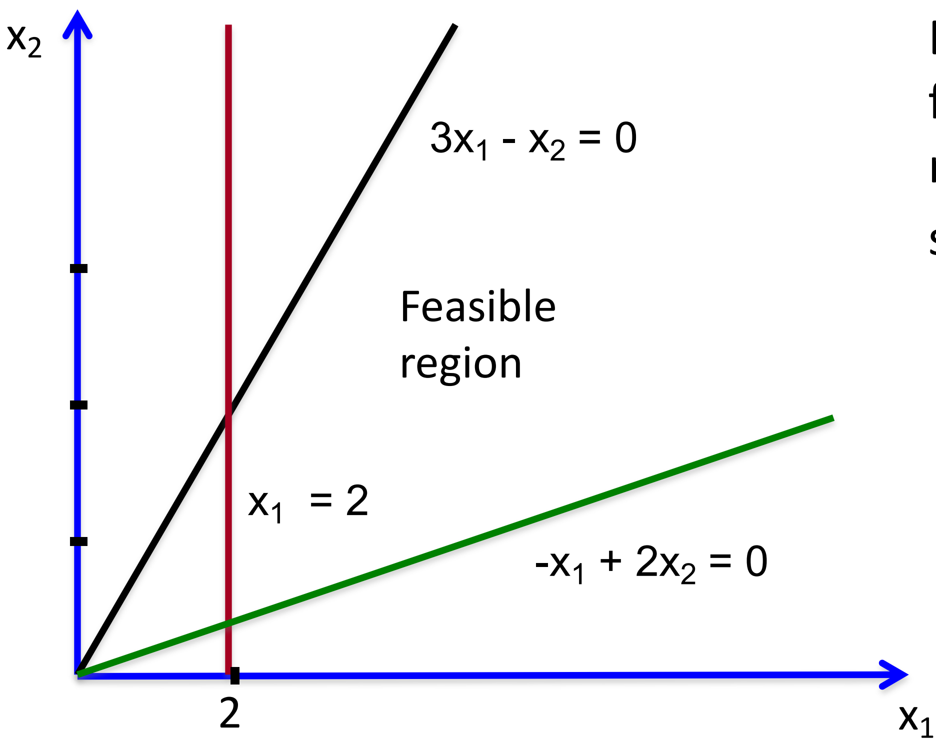
Observations:

- In 2 dimensions, the feasible region is a polygon
- We stop only when the dashed line intersects the feasible region in a corner point of the polygon
  - Or in degenerate cases, when the line coincides with one of the sides of the polygon
- How can we compute the values of x_1, x_2 when we stop?
  - A corner point is the intersection of 2 sides, hence they satisfy 2 constraints with equality
- In Example 2, we stop at Z=36
- The solution of
  - $2x_2 = 12$
  - $3x_1 + 2x_2 = 18$
- Hence, $x_1 = 2$, $x_2 = 6$

The Graphical Method

Can the graphical method keep going without ever terminating?

- YES, when the polyhedron is unbounded
- But if this happens, the optimal solution is $+\infty$



Example of an unbounded feasible region:

$\max Z = 4x_1 + 2x_2$

s.t.

$x_1 \geq 2$

$3x_1 - x_2 \geq 0$

$-x_1 + 2x_2 \geq 0$

$x_1, x_2 \geq 0$

The Graphical Method

Insights gained from the graphical method:

- If an optimal solution exists, it is attained at a corner point of the polygon
- In higher dimensions:
  - the feasible region is a polyhedron
  - Again, it suffices to look at the corner points of the polyhedron
  - Till 3 dimensions, we can do this geometrically
  - When $n \geq 4$, we should do it algebraically

- Idea for higher dimensional problems: Try to examine corner points of the polyhedron till we reach the optimal one

The Graphical Method

- Q: Could we enumerate all corner point solutions and pick the best one?
  - Not an efficient algorithm, polyhedra can have exponentially many corner points.
- BUT: We can try to think of a more clever way to search for the best corner point
  - Essentially what simplex does (next lecture)