1. *(2 points)* Identify the locations of saddle points and extrema of the following function, defined in $\mathbb{R}^2$:

$$f(x, y) = x^3 + y^3 - 2xy.$$

2. *(2 points)* Find the extrema, their type (minima or maxima), and associated Lagrange multipliers λ, of the function $f(x, y) = x^3 + y^3 - 2xy$ under the constraint $x + y = 10$.

3. *(1 point)* Identify the locations of saddle points and extrema of the following function, defined in $\mathbb{R}^2$:

$$f(x, y) = \frac{1}{1 + e^{(x^2+y^2-1)^2}}.$$

4. *(2 points)* Use the Gram-Schmidt process to calculate a set of orthonormal vectors using the following vectors *in the given order*:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

5. *(2 points)* Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

6. *(1 point)* Find the inverse of matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

1. *(2 points)* Identify the locations of saddle points and extrema of the following function, defined in $\mathbb{R}^2$:

$$f(x,y) = x^3 + y^3 - 2xy.$$

Solution: We note that

$$f_x(x,y) = 3x^2 - 2y, \quad f_y(x,y) = 3y^2 - 2x, \quad f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -2.$$

Since

$$f_x(x,y) = 0 \Leftrightarrow 3x^2 - 2y = 0 \Leftrightarrow y = \frac{3}{2}x^2$$
$$f_y(x,y) = 0 \Leftrightarrow 3y^2 - 2x = 0 \Leftrightarrow x = \frac{3}{2}y^2,$$

it follows that

$$x = \frac{3}{2}\left(\frac{3}{2}\right)^2 x^4 \Leftrightarrow x\left(\left(\frac{3x}{2}\right)^3 - 1\right) = 0,$$

therefore $x = y = 0$ or $x = y = \frac{2}{3}$.
The case $x = y = \frac{2}{3}$ is more straightforward, because we then have

$$D = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 36\left(\frac{2}{3}\right)^2 - (-2)^2 = 12 > 0,$$

και $f_{xx}(x,y) = 4$, therefore that point is a local minimum.

The case $x = y = 0$ corresponds to $D < 0$, and therefore that point is a saddle point.

2. *(2 points)* Find the extrema, their type (minima or maxima), and associated Lagrange multipliers λ, of the function $f(x,y) = x^3 + y^3 - 2xy$ under the constraint $x + y = 10$.

Solution: We require

$$\nabla f = \lambda \nabla g \Leftrightarrow (3x^2 - 2y, 3y^2 - 2x) = (\lambda, \lambda),$$

therefore

$$3x^2 - 2y = 3y^2 - 2x \Leftrightarrow 3(x-y)(x+y) = 2(y-x) \Leftrightarrow (x-y)[3(x+y)+2] = 0 \Leftrightarrow x = y = 5, \lambda = 65.$$

To determine the type of the extremum, we can substitute the constraint into the objective function, resulting in the one dimensional function

$$x^3 + (10-x)^3 - 2x(10-x) = x^3 + 1000 - 300x + 30x^2 - x^3 - 20x + 2x^2 = 32x^2 - 320x + 1000,$$

which is a quadratic function whose quadratic term has a positive coefficient, therefore the extremum is actually a minimum.

3. *(1 point)* Identify the locations of saddle points and extrema of the following function, defined in $\mathbb{R}^2$:

$$f(x,y) = \frac{1}{1 + e^{(x^2+y^2-1)^2}}.$$

Solution: Observe that the function is actually a function of the square of the distnace of the point (x, y) from the origin. Therefore, the function is symmetric with respect with the origin.

Note that the polynomial $(1+(d-1))^2$ is minimized at $d=1$, therefore the given function attains a local maximum in the circle $x^2+y^2=1$.

Also, the polynomial $(1+(d-1))^2$ is maximized at $d=0$, given that we must have $d>0$, therefore the function attains a local minimum at the origin.

Finally, the polynomial $(1+(d-1))^2$ goes to infinity ad $d\to\infty$, therefore the given function goes to zero as we move away from the origin.

4. *(2 points)* Use the Gram-Schmidt process to calculate a set of orthonormal vectors using the following vectors *in the given order*:

$$\mathbf{v}_1=\begin{bmatrix}2\\0\\-2\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}0\\0\\2\end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Solution:

$$\begin{aligned}
e_1 &= \frac{v_1}{\|v_1\|}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix},\\
v_2\cdot e_1 &= -\sqrt{2},\\
q_2 &= \begin{bmatrix}0\\0\\2\end{bmatrix}+\sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}\\
e_2 &= \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}\\
v_3\cdot e_1 &= \frac{1}{\sqrt{2}},\\
v_3\cdot e_2 &= \frac{1}{\sqrt{2}},\\
q_3 &= \begin{bmatrix}1\\1\\0\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix},\\
e_3 &= \begin{bmatrix}0\\1\\0\end{bmatrix}.
\end{aligned}$$

■

5. *(2 points)* Find the eigenvalues and eigenvectors of the matrix

$$A=\begin{bmatrix}1&1&1\\0&1&0\\1&1&1\end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\begin{vmatrix}1-\lambda&1&1\\0&1-\lambda&0\\1&1&1-\lambda\end{vmatrix}=0\Leftrightarrow(1-\lambda)[(1-\lambda)^2-1]=(1-\lambda)(1+\lambda^2-2\lambda-1)=\lambda(1-\lambda)(\lambda-2),$$

therefore the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$, and

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To find the eigenvectors corresponding to the eigenvalue $\lambda_1 = 2$, we write

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = 0, \quad x_1 = x_3,$$

We select the vector $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$.

To find the eigenvectors corresponding to the eigenvalue $\lambda_2 = 1$, we write

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 + x_3 = 0, \quad x_1 + x_2 = 0,$$

We select the vector $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$.

To find the eigenvectors corresponding to the eigenvalue $\lambda_3 = 0$, we write

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = 0, \ x_1 + x_2 + x_3 = 0$$

We select the vector $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$. ■

6. *(1 point)* Find the inverse of matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Solution: We can perform Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \qquad (R2 = -R2, \ \ R3 = R3 - R1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \qquad (R3 = -\frac{1}{2}R3)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad (R1 = R1 - R2 - R3)$$

Therefore,

$$S^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

■